Math 1210 Test 3. Spring 2016

1. Write each of the following compositions as an equivalent *algebraic* expression (in terms of u).

(a) $\sin(\cos^{-1}(u))$

(b) $\sin(2\cos^{-1}(u))$

2. If $0 < \alpha < \pi/2$ and $\sin\alpha = 3/5$, what is $\sin(2\alpha)$?

3. Find the exact values of the following expressions. Simplify your answer.

(a) $\tan(15°)$

(b) $\sin\left(\cos^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{3}\right)\right)$

(c) $\sin(12°)\cos(18°) + \cos(12°)\sin(18°)$

4. Find all solutions to the following equations in the interval $[0, 2\pi]$.

(a) $2\cos\theta + 1 = 0$

(b) $2\cos^2\theta + \cos\theta - 1 = 0$

5. Give a general formula for all the solutions to the following equations.

(a) $\sin(3\theta) = -1$

(b) $4(1 + \sin\theta) = \cos^2\theta$

6. Verify each of the following identities. Be sure to clearly show all of your steps.

(a) $\cos x \sec x - \cos^2 x = \sin^2 x$

(b) $\dfrac{\sec\theta + \tan\theta}{\cot\theta + \cos\theta} = \tan\theta \cdot \sec\theta$

(c) $\dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta} = 2\sec^2\theta$

(d) $\dfrac{\sin(4\theta) + \sin(8\theta)}{\cos(4\theta) + \cos(8\theta)} = \tan(6\theta)$

SOLUTIONS

1. Put $\theta = \cos^{-1}(u)$, so $\cos\theta = u$. Then

$$(a)\quad \sin(\cos^{-1}(u)) = \sin\theta = \sqrt{1-u^2}$$

$$\begin{aligned}(b)\quad \sin(2\cos^{-1}(u)) &= \sin 2\theta \\ &= 2\sin\theta\cos\theta = 2u\sqrt{1-u^2}\end{aligned}$$

2. $\sin\alpha = 3/5 \implies \cos\alpha = 4/5$ (Pythagorean theorem). So

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$$

3. (a)

$$\begin{aligned}\tan 15^\circ = \tan(30^\circ/2) &= \frac{\sin 30^\circ}{1+\cos 30^\circ} \\ &= \frac{1/2}{1+\sqrt{3}/2} = \frac{1}{2+\sqrt{3}}\end{aligned}$$

(b) Put $A = \cos^{-1}(1/2)$ so $\cos A = 1/2$, and put $B = \cos^{-1}(1/3)$ so $\cos B = 1/3$. Then $\sin A = \sqrt{3}/2$ and $\sin B = \sqrt{8}/3$, so

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B \\ &= \frac{\sqrt{3}}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{\sqrt{8}}{3} = \frac{\sqrt{3}+\sqrt{8}}{6}.\end{aligned}$$

(c)

$$\begin{aligned}&\sin(12^\circ)\cos(18^\circ) + \cos(12^\circ)\sin(18^\circ) \\ &\quad = \sin(12^\circ + 18^\circ) = \sin(30^\circ) = 1/2.\end{aligned}$$

4. (a)

$$\begin{aligned}2\cos\theta + 1 &= 0 \\ \cos\theta &= -1/2 \\ \theta &= 2\pi/3,\ 4\pi/3\end{aligned}$$

(b) Let $u = \cos\theta$ to get

$$\begin{aligned}2u^2 + u - 1 &= 0 \\ (2u-1)(u+1) &= 0 \\ u &= 1/2, u = -1\end{aligned}$$

Then $\cos\theta = 1/2$, meaning $\theta = \pi/3$ or $\pi/6$, or $\cos\theta = -1$, meaning $\theta = \pi$.

5. (a)

$$\begin{aligned}\sin(3\theta) &= -1 \\ 3\theta &= \frac{3\pi}{2} + 2\pi n \\ \theta &= \frac{\pi}{2} + \frac{2\pi n}{3}\end{aligned}$$

(b)

$$\begin{aligned}4 + 4\sin\theta &= 1 - \sin^2\theta \\ \sin^2\theta + 4\sin\theta + 3 &= 0 \\ (\sin\theta + 3)(\sin\theta + 1) &= 0\end{aligned}$$

Either $\sin\theta = -3$, which is not possible, or $\sin\theta = -1$, so that $\theta = 3\pi/2 + 2\pi n$.

6. (a)

$$\begin{aligned}\cos x\sec x - \cos^2 x &= \cos x\cdot\frac{1}{\cos x} - \cos^2 x \\ &= 1 - \cos^2 x \\ &= \sin^2 x\end{aligned}$$

(b)

$$\begin{aligned}\frac{\sec\theta + \tan\theta}{\cot\theta + \cos\theta} &= \frac{\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\sin\theta} + \cos\theta} \\ &= \frac{\frac{1+\sin\theta}{\cos\theta}}{\frac{\cos\theta + \cos\theta\sin\theta}{\sin\theta}} \\ &= \frac{1+\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta + \cos\theta\sin\theta} \\ &= \frac{(1+\sin\theta)\sin\theta}{\cos^2\theta(1+\sin\theta)} \\ &= \frac{\sin\theta}{\cos^2\theta} \\ &= \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta} \\ &= \tan\theta\sec\theta\end{aligned}$$

(c)

$$\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}$$
$$=\frac{1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)}+\frac{1-\sin\theta}{(1+\sin\theta)(1-\sin\theta)}$$
$$=\frac{2}{(1-\sin\theta)(1+\sin\theta)}$$
$$=\frac{2}{1-\sin^2\theta}$$
$$=\frac{2}{\cos^2\theta}$$
$$=2\sec^2\theta$$

(d)

$$\frac{\sin(4\theta)+\sin(8\theta)}{\cos(4\theta)+\cos(8\theta)}$$
$$=\frac{2\sin((4\theta+8\theta)/2)\cos((4\theta-8\theta)/2)}{2\cos((4\theta+8\theta)/2)\cos((4\theta-8\theta)/2)}$$
$$=\frac{2\sin(6\theta)\cos(-2\theta)}{2\cos(6\theta)\cos(-2\theta)}$$
$$=\frac{\sin(6\theta)}{\cos(6\theta)}$$
$$=\tan(6\theta)$$