Math 2040 Test 3. Fall 2016

1. Use implicit differentiation to find dy/dx when

$$\cos(xy) = x + 1.$$

2. Use implicit differentiation to find the slope of the tangent line to the curve

$$x^3 + 2xy^3 + 3y^2 = 6$$

at the point $(1, 1)$.

3. Use logarithmic differentiation to find the derivative of the functions.

(a) $f(x) = \dfrac{(x+1)^2(x+2)^3}{(x+3)^4(x+4)^5}$

(b) $f(x) = (1 + x)^{1/x}$

4. Compute the derivatives of the functions. Simplify your answers.

(a) $f(x) = 7^{3x^2+5}$

(b) $f(x) = \ln\left(\dfrac{x}{x+1}\right)$

(c) $f(x) = \dfrac{\sinh x - 3}{\cosh x + 1}$

(d) $f(x) = x\cos^{-1}(x^3)$

5. An object is launched vertically with an initial velocity of 96 ft/sec. Its height as a function of time is given by the formula:

$$p(t) = -16t^2 + 96t$$

What is the maximum height the object will reach?

6. The population of a metropolitan area has grown from 100,000 to 130,000 over the past twenty years. Assuming that the population is growing exponentially, what will the population be in twenty more years?

7. A spherical balloon is inflated. Air is pumped into the balloon at a rate of $0.5\, in^3/min$. How fast is the radius of the balloon increasing when the radius is $4\, in$? Recall that the volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$.

8. Two horses with riders depart from the same point. The first horse and rider heads due north at a rate of $10\, mph$. The second heads due west at a rate of $15\, mph$. How fast is the distance between them changing after 12 minutes ($1/5\, hr$)? 9. List all the critical points of the function $f(x) = \dfrac{x-1}{x^2+3}$.

10. Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 2$ on the interval $[0, 4]$.

SOLUTIONS

1.

$$
\begin{aligned}
-\sin(xy)(xy' + y) &= 1 \\
-xy'\sin(xy) - y\sin(xy) &= 1 \\
-xy'\sin(xy) &= 1 + y\sin(xy) \\
y' &= \frac{1 + y\sin(xy)}{-x\sin(xy)}
\end{aligned}
$$

2.

$$
\begin{aligned}
3x^2 + 2y^3 + 6xy^2y' + 6yy' &= 0 \\
(6xy^2 + 6y)y' &= -3x^2 - 2y^3 \\
y' &= \frac{-3x^2 - 2y^3}{6xy^2 + 6y}
\end{aligned}
$$

The slope of the tangent line at $(1, 1)$ is

$$m = y'(1,1) = \frac{-3-2}{6+6} = -\frac{5}{12}.$$

3.

$$
\begin{aligned}
\ln(f(x)) &= \ln\left(\frac{(x+1)^2(x+2)^3}{(x+3)^4(x+4)^5}\right) \\
&= 2\ln(x+1) + 3\ln(x+2) \\
&\quad - 4\ln(x+3) - 5\ln(x+5)
\end{aligned}
$$

so

$$\frac{1}{f(x)}f'(x) = \frac{2}{x+1} + \frac{3}{x+2} - \frac{4}{x+3} - \frac{5}{x+4}$$

Multiply both sides by $f(x)$:

$$
\begin{aligned}
f'(x) = \frac{(x+1)^2(x+2)^3}{(x+3)^4(x+4)^5}\Bigg(&\frac{2}{x+1} + \\
&+ \frac{3}{x+2} - \frac{4}{x+3} - \frac{5}{x+4}\Bigg).
\end{aligned}
$$

4. (a)

$$f'(x) = 7^{3x^2+5} \cdot \ln 7 \cdot 6x$$

(b) Rewrite $f(x) = \ln x - \ln(x+1)$. Then

$$
\begin{aligned}
f'(x) &= \frac{1}{x} - \frac{1}{x+1} \\
&= \frac{(x+1) - x}{x(x+1)} \\
&= \frac{1}{x(x+1)}
\end{aligned}
$$

(c)

$$
\begin{aligned}
f'(x) &= \frac{(\cosh x + 1)\cosh x - (\sin x - 3)\sinh x}{(\cosh x + 1)^2} \\
&= \frac{\cosh^2 x + \cosh x - \sinh^2 x + 3\sinh x}{(\cosh x + 1)^2} \\
&= \frac{1 + \cosh x + 3\sinh x}{(\cosh x + 1)^2}
\end{aligned}
$$

(d)

$$
\begin{aligned}
f'(x) &= x\left(-\frac{1}{\sqrt{1-x^6}} \cdot 3x^2\right) + \cos^{-1}(x^3) \\
&= -\frac{3x^3}{\sqrt{1-x^6}} + \cos^{-1}(x^3)
\end{aligned}
$$

5. The maximum height is when $p'(t) = 0$, so when

$$
\begin{aligned}
-32t + 96 &= 0 \\
32t &= 96 \\
t &= 3
\end{aligned}
$$

The height at that time is

$$p(3) = -16 \cdot 3^2 + 96 \cdot 3 = 144 ft.$$

6. Use $A = A_0e^{rt}$. First find the rate of growth:

$$130 = 100e^{20r} \implies r = \frac{\ln(130/100)}{20} \approx 0.013.$$

In twenty more years,

$$A = 100e^{0.013 \cdot 40} = 168.2.$$

The population will be around 168,000.

7.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt} \\ 0.5 &= 4\pi 4^2 \frac{dr}{dt} \\ \frac{1}{128\pi} &= \frac{dr}{dt} \end{aligned}$$

8. Let y be the distance the first rider has covered, x be the distance the second has covered, and D be distance between them. Then $dx/dt = 15$, $dy/dt = 10$, and by the Pythagorean theorem

$$\begin{aligned} x^2 + y^2 &= D^2 \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 2D\frac{dD}{dt} \\ x\frac{dx}{dt} + y\frac{dy}{dt} &= D\frac{dD}{dt} \end{aligned}$$

In 12 minutes, $y = 2$, $x = 3$, and so $D = \sqrt{13}$. Then

$$3 \cdot 15 + 2 \cdot 10 = \sqrt{13}\frac{dD}{dt}$$

so $dD/dt = 65/\sqrt{13}$.

9. The function is defined for all real numbers. Its derivative is

$$\begin{aligned} f'(x) &= \frac{(x^2+3)(1) - (x-2)(2x)}{(x^2+3)^2} \\ &= \frac{-x^2 + 2x + 3}{(x^2+3)^2} \end{aligned}$$

The derivative is defined for all real numbers. Therefore the only critical points are where $f'(x) = 0$, when

$$\begin{aligned} x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x = 3, \quad x &= -1 \end{aligned}$$

10. The derivative is zero when

$$\begin{aligned} 3x^2 - 6x &= 0 \\ 3x(x-2) &= 0 \\ x = 0, \quad x &= 2 \end{aligned}$$

Then

$$\begin{aligned} f(0) &= 2 \\ f(2) &= -2 \\ f(4) &= 18 \end{aligned}$$

The global minimum is -2 when $x = 2$. The global maximum is 18 when $x = 4$.