Math 2040 Test 4. Fall 2016

1. Explain why $f(x) = x^4 - 5x + c$ has at one most one root in the interval $[-1, 1]$.

2. Each part is worth four points. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{2x^3 + 7x + 5}{4x^3 + 2x^2 - 1}$

(b) $\displaystyle\lim_{x\to-\infty} \frac{x^2 + 2x + 1}{\sqrt[3]{8x^6 + 5x}}$

(c) $\displaystyle\lim_{x\to\infty} x^2 e^{-x}$

(d) $\displaystyle\lim_{x\to 0} \frac{\sin x \cos x}{x}$

(e) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^{2x}$

3. Each part is worth four points. $f(x) = \dfrac{\ln x}{x^2}$

(a) What is the domain of $f(x)$?

(b) List the x and/or y intercepts of $f(x)$.

(c) Give the equations of any horizontal or vertical asymptotes.

(d) Compute and simplify $f'(x)$.

(e) What are the critical points of $f(x)$?

(f) On which intervals is $f(x)$ increasing?

(g) Compute and simplify $f''(x)$.

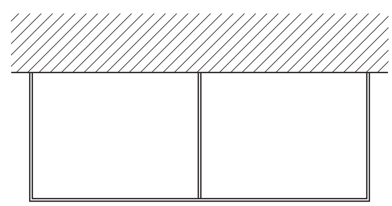
(h) What are the inflection points of $f(x)$?

(i) On which intervals is $f(x)$ concave up?

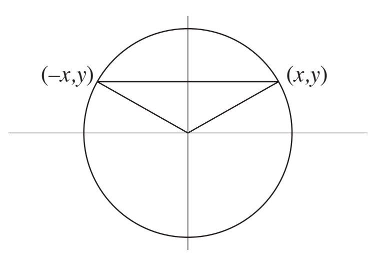
(j) Use the information gathered in the previous parts to draw the graph of $f(x)$.

4. Find the point on the line $y = 3x + 4$ that is closest to the origin.

5. A rectangular plot of land with a natural boundary on the north side is to be fenced in on the remaining three sides. Additionally, fencing is to be used to divide the plot in half, as shown in the figure. With 400 feet of fencing, what is the largest area that can be enclosed?



6. An isosceles triangle has one of its corners at the origin and the other two, (x, y) and $(-x, y)$, on the top half of the unit circle. What should x be in order to maximize the area of the triangle?



1. f is a polynomial, hence differentiable and continuous. By Rolle's theorem, if f had two roots in the interval, then there would be a value c between them where $f'(c) = 0$. However, $f'(x) = 4x^3 - 5$, so it is only zero when $x = \pm\sqrt{5/4}$. Both of these fall outside the interval.

2. (a)

$$\lim_{x\to\infty} \frac{2x^3 + 7x + 5}{4x^3 + 2x^2 - 1} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to\infty} \frac{2 + 7/x^2 + 5/x^3}{4 + 2/x - 1/x^3} = \frac{1}{2}$$

(b)

$$\lim_{x\to-\infty} \frac{x^2 + 2x + 1}{\sqrt[3]{8x^6 + 5x}} \cdot \frac{1/x^2}{1/\sqrt[3]{x^6}} = \lim_{x\to-\infty} \frac{1 + 2/x + 1/x^2}{\sqrt[3]{8 + 5/x^5}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$$

(c)

$$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

(d)

$$\lim_{x\to 0} \frac{\sin x \cos x}{x} \overset{H}{=} \lim_{x\to 0} \frac{-\sin^2 x + \cos^2 x}{1} = 1$$

(e) Let $y = \lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^{2x}$. Then

$$\begin{aligned}
\ln y &= \lim_{x\to\infty} \ln\left(1 + \frac{3}{x}\right)^{2x} \\
&= \lim_{x\to\infty} 2x \ln\left(1 + \frac{3}{x}\right) \\
&= \lim_{x\to\infty} \frac{2\ln(1 + 3x^{-1})}{x^{-1}} \\
&\overset{H}{=} \lim_{x\to\infty} \frac{2 \cdot \frac{1}{1+3x^{-1}} \cdot (-3x^{-2})}{-x^-2} \\
&= \lim_{x\to\infty} \frac{6}{1 + 3/x} \\
&= 6,
\end{aligned}$$

so $y = e^6$.

3. (a) The domain is $(0, \infty)$.

(b) There is no y-intercept. x-intercept:

$$f(x) = 0 : \ln x = 0 : x = e^0 = 1.$$

(c) There is a vertical asymptote at $x = 0$ (note $\lim_{x\to 0} \ln x/x^2 = -\infty$).

$$\lim_{x\to\infty} \frac{\ln x}{x^2} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

so the function has horizontal asymptote $y = 0$.

(d)

$$\begin{aligned}
f'(x) &= \frac{x^2 \cdot 1/x - \ln x \cdot 2x}{x^4} \\
&= \frac{x - 2x\ln x}{x^4} \\
&= \frac{1 - 2\ln x}{x^3}.
\end{aligned}$$

(e)

$$\begin{aligned}
f'(x) &= 0 \\
1 - 2\ln x &= 0 \\
\ln x &= 1/2x = e^{1/2}
\end{aligned}$$

(f) Make a sign chart. Note that $1 - 2\ln x$ is positive when $0 < x < e^{1/2}$, and negative when $x > e^{1/2}$. x^3 is positive for all positive x. So $f(x)$ is increasing on the interval $(0, e^{1/2})$.

(g)

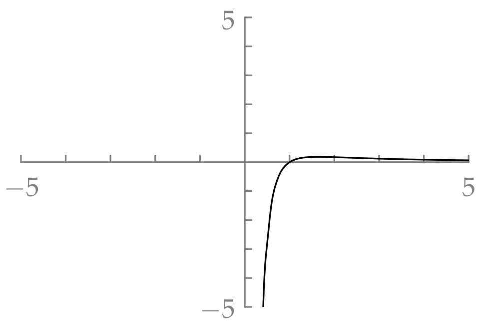
$$\begin{aligned}
f''(x) &= \frac{x^3(-2/x) - (1 - 2\ln x)3x^2}{x^6} \\
&= \frac{-2x^2 - 3x^2 + 6\ln x \cdot x^2}{x^6} \\
&= \frac{-5 + 6\ln x}{x^4}
\end{aligned}$$

(h)

$$\begin{aligned} f''(x) &= 0 \\ -5 + 6\ln x &= 0 \\ \ln x &= 5/6 \\ x &= e^{5/6} \end{aligned}$$

(i) Make a sign chart. Note that $-5+6\ln x$ is positive when $x > e^{5/6}$ and negative when $x < e^{5/6}$. The x^4 term is always positive. Therefore $f(x)$ is concave up on the interval $(e^{5/6}, \infty)$.

(j)



4. Minimize

$$\begin{aligned} D &= \sqrt{x^2+y^2} \\ &= \sqrt{x^2+(3x+4)^2} \\ &= \sqrt{10x^2+24x+16}. \end{aligned}$$

The derivative is

$$D'(x) = \frac{20x+24}{2\sqrt{10x^2+24x+16}}.$$

The minimum is when $D' = 0$:

$$D' = 0 \implies x = -6/5 \implies y = 2/5.$$

5. Maximize the area $A = xy$. The constraint is that $x + 3y = 400 \implies x = 400 - 3y$. Then

$$A = (400-3y)y = 400y - 3y^2.$$

Then $A' = 400 - 6y$. The critical point is when $A' = 0$, so $y = 200/3$, and then $x = 200$. The maximum area is $A = 200 \cdot 200/3 = 40000/3$ square feet.

6. The base is $2x$ and the height is y, so we need to maximize $A = xy$ subject to the constraint $x^2 + y^2 = 1$. Then

$$A = x\sqrt{1-x^2}$$

so

$$\begin{aligned} A' &= x \cdot \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) + (1-x^2)^{1/2} \\ &= \frac{-x^2}{(1-x^2)^{1/2}} + (1-x^2)^{1/2} \\ &= \frac{-x^2+(1-x^2)}{(1-x^2)^{1/2}} \\ &= \frac{1-2x^2}{(1-x^2)^{1/2}} \end{aligned}$$

The maximum is when $A' = 0$:

$$1 - 2x^2 = 0 \implies x = 1/\sqrt{2}.$$