Math 2050 Test 1. Spring 2016

1. Find the most general antiderivative of each function.

(a) $f(x) = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{\sqrt{x}}$

(b) $f(x) = (x+3)^2$

(c) $f(x) = \cos x + \frac{1}{\cos^2 x}$

2. Find the function $f(x)$ which satisfies the following conditions

$f''(x) = 4x^3+6x^2+6x+4,\ f(0) = 4,\ f'(1) = 13.$

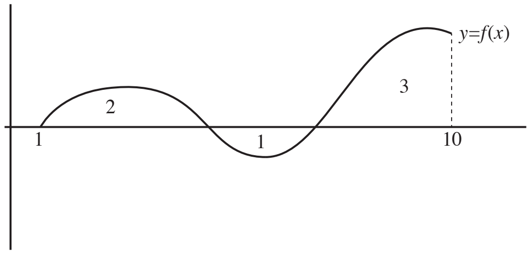
3. Evaluate the integral $\int_0^2 \sqrt{4-x^2}\,dx$ by interpreting it in terms of area.

4. Use the fundamental theorem of calculus to evaluate the following expressions.

(a) $\frac{d}{dx}\int_1^x \sin(t^6)\,dt$

(b) $\frac{d}{dx}\int_1^{x^2} \sin(t^3)\,dt$

5. The areas of three regions between the graph of $y = f(x)$ and the x-axis are shown below. What is $\int_1^{10} f(x)\,dx$?



6. Evaluate the definite integrals.

(a) $\int_0^3 x(x+1)\,dx$

(b) $\int_0^{\pi/4} (\sin x + \cos x)\,dx$

7. Evaluate the indefinite integrals (you may or may not need to use a substitution).

(a) $\int \frac{x^2+1}{x}\,dx$

(b) $\int e^{7x}\,dx$

(c) $\int e^x \sin(e^x)\,dx$

(d) $\int \cos^5 x \sin x\,dx$

(e) $\int x\sqrt{x^2+11}\,dx$

(f) $\int x(x+3)^{25}\,dx$

8. Use a Riemann sum to calculate the integral

$$\int_1^3 (x^2-5)\,dx.$$

No credit will be given for any other method.

SOLUTIONS

1.

$$
\begin{aligned}
(a)\quad f(x) &= \frac{1}{x} + x^{-2} + x^{-1/2} \\
F(x) &= \ln|x| - x^{-1} + 2x^{1/2} + C \\
&= \ln|x| - \frac{1}{x} + 2\sqrt{x} + C \\
(b)\quad f(x) &= x^2 + 6x + 9 \\
F(x) &= \frac{1}{3}x^3 + 3x^2 + 9x + C \\
(c)\quad f(x) &= \cos x + \sec^2 x \\
F(x) &= \sin x + \tan x + C
\end{aligned}
$$

2. The first antiderivative:

$$f'(x) = x^4 + 2x^3 + 3x^2 + 4x + C$$

Since $f'(1) = 13$,

$$13 = 1 + 2 + 3 + 4 + C \implies C = 3.$$

The second antiderivative:

$$f(x) = \frac{1}{5}x^5 + \frac{1}{2}x^4 + x^3 + 2x^2 + 3x + D$$

Since $f'(0) = 4$, $D = 4$. Therefore

$$f(x) = \frac{1}{5}x^5 + \frac{1}{2}x^4 + x^3 + 2x^2 + 3x + 4.$$

3. The function $y = \sqrt{4 - x^2}$ is the top half of a circle of radius two. The integral in question is the area of the portion of the circle in the first quadrant. So

$$\int_0^2 \sqrt{4 - x^2}\,dx = \frac{1}{4}(4\pi) = \pi.$$

4.

(a) $$\frac{d}{dx}\int_1^x \sin(t^6)\,dt = \sin(x^6)$$

(b) $$\frac{d}{dx}\int_1^{x^2} \sin(t^3)\,dt = \sin(x^6)\cdot 2x$$

5.

$$\int_1^{10} f(x)\,dx = 2 - 1 + 3 = 4.$$

6.

$$
\begin{aligned}
(a)\quad \int_0^3 x(x+1)\,dx &= \int_0^3 x^2 + x\,dx \\
&= \frac{1}{3}x^3 + \frac{1}{2}x^2\Big|_0^3 \\
&= (9 + 9/2) - 0 = 27/2.
\end{aligned}
$$

$$
\begin{aligned}
(b)\quad &\int_0^{\pi/4} (\sin x + \cos x)\,dx \\
&= (-\cos x + \sin x)\Big|_0^{\pi/4} \\
&= \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (-1 + 0) \\
&= 1.
\end{aligned}
$$

7. (a)

$$
\begin{aligned}
\int \frac{x^2+1}{x}\,dx &= \int \left(x + \frac{1}{x}\right)\,dx \\
&= \frac{1}{2}x^2 + \ln|x| + C
\end{aligned}
$$

(b) Let $u = 7x$, so that $du = 7dx$. Then

$$\int e^{7x}\,dx = \int \frac{1}{7}e^u\,du = \frac{1}{7}e^u + C = \frac{e^{7x}}{7} + C.$$

(c) Let $u = e^x$, so that $du = e^x dx$. Then

$$
\begin{aligned}
\int e^x \sin(e^x)\,dx &= \int \sin u\,du \\
&= -\cos u + C \\
&= -\cos(e^x) + C.
\end{aligned}
$$

(d) Let $u = \cos x$, so that $du = -\sin x\,dx$. Then

$$
\begin{aligned}
\int \cos^5 x \sin x\,dx &= \int u^5 \cdot (-1du) \\
&= -\frac{u^6}{6} + C \\
&= -\frac{\cos^6 x}{6} + C.
\end{aligned}
$$

(e) Let $u = x^2 + 11$ so that $du = 2xdx$. Then

$$\begin{aligned}\int x\sqrt{x^2+11}\,dx &= \int \frac{1}{2}\sqrt{u}\,du \\ &= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C \\ &= \frac{1}{3}(x^2+11)^{3/2} + C\end{aligned}$$

(f) Let $u = x + 3$ so that $du = dx$. Then

$$\begin{aligned}\int x(x+3)^{25}\,dx &= \int (u-3)u^{25}\,du \\ &= \frac{u^{27}}{27} - \frac{3u^{26}}{26} + C \\ &= \frac{(x+3)^{27}}{27} - \frac{3(x+3)^{26}}{26} + C\end{aligned}$$

8. $\Delta x = 2/n$ and $x_i = 1 + 2i/n$, so

$$f(x_i) = \left(1 + \frac{2i}{n}\right)^2 - 5 = -4 + \frac{4i}{n} + \frac{4i^2}{n^2}.$$

Then

$$\begin{aligned}&\int_1^3 (x^2 - 5)\,dx \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-4 + \frac{4i}{n} + \frac{4i^2}{n^2}\right)\frac{2}{n} \\ &= \lim_{n\to\infty}\left(-\frac{8}{n}\sum_{i=1}^{n} + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2\right) \\ &= \lim_{n\to\infty}\left(-\frac{8}{n}\cdot n + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\ &= \lim_{n\to\infty}\left(-8 + 4\cdot\frac{n+1}{n} + \frac{4}{3}\cdot\frac{2n^2+3n+1}{n^2}\right) \\ &= -8 + 4 + \frac{8}{3} \\ &= -4/3.\end{aligned}$$