Math 2050 Test 3. Spring 2016

1. Use the method of *integration by parts* to evaluate:

$$\int x^3 \ln x \, dx.$$

2. Use the method of *trigonometric substitution* to evaluate:

$$\int \frac{dx}{\sqrt{9 - x^2}}.$$

3. Use the method of *partial fractions* to evaluate:

$$\int \frac{5x + 6}{x^2 + x - 12} \, dx.$$

4. Evaluate the improper integral. If it diverges say so:

$$\int_0^\infty x e^{-x} \, dx.$$

5. Evaluate the improper integral. If it diverges say so:

$$\int_0^2 \frac{1}{(x-2)^3} \, dx.$$

Questions 6–10. Evaluate the integrals.

6. $\displaystyle\int \sin^5 x \cos^3 x \, dx$

7. $\displaystyle\int x^3 \sqrt{x^2 + 1} \, dx$

8. $\displaystyle\int \frac{5x^2 + 2}{x^3 + x} \, dx$

9. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 2x + 10}}$

10. $\displaystyle\int x^3 \sin(x^2) \, dx$

1. Let $u = \ln x$ and $dv = x^3$. Then $du = (1/x)dx$ and $v = x^4/4$, so

$$\begin{aligned}\int x^3 \ln x\,dx &= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C\end{aligned}$$

2. Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$, so

$$\begin{aligned}\int \frac{dx}{\sqrt{9-x^2}} &= \int \frac{3\cos\theta\,d\theta}{3\cos\theta} \\ &= \int d\theta \\ &= \theta + C \\ &= \sin^{-1}(x/3) + C\end{aligned}$$

3. Write

$$\frac{5x-6}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}.$$

Then

$$5x+6 = A(x-3) + B(x+4).$$

Substitute $x = 3$ to find that $B = 3$ and $x = -4$ to find that $A = 2$. Therefore

$$\begin{aligned}\int \frac{5x+6}{x^2+x-12}\,dx &= \int \left(\frac{2}{x+4} + \frac{3}{x-3}\right)dx \\ &= 2\ln|x+4| + 3\ln|x-3| + C\end{aligned}$$

4. First evaluate the integral using partial fractions. Put $u = x$, and $dv = e^{-x}dx$ so that $du = dx$ and $v = -e^{-x}$. Then

$$\begin{aligned}\int xe^{-x}\,dx &= -xe^{-x} + \int e^{-x}\,dx \\ &= -xe^{-x} - e^{-x} + C \\ &= \frac{-x-1}{e^x} + C.\end{aligned}$$

So

$$\begin{aligned}\int_{[}^{\infty} xe^{-x}\,dx &= \lim a \to \infty \frac{-x-1}{e^x}\bigg|_0^a \\ &= \lim_{a\to\infty} \frac{-a-1}{e^a} + \frac{1}{e^0} \\ &\overset{H}{=} \lim_{a\to\infty} \frac{-1}{e^a} + 1 \\ &= 1.\end{aligned}$$

5.

$$\begin{aligned}\int_0^2 \frac{1}{x-2}^3\,dx &= \lim_{a\to 2^-} \int_0^a \frac{1}{(x-2)^3}\,dx \\ &= \lim_{a\to 2^-} -\frac{1}{2(x-2)^2}\bigg|_0^a \\ &= \lim_{a\to 2^-} \left(-\frac{1}{2(a-2)^2} + \frac{1}{8}\right) \\ &= -\infty\end{aligned}$$

The integral diverges.

6.

$$\int \sin^5 x \cos^3 x\,dx = \int \sin^5 x(1-\sin^2 x)\cos x\,dx$$

Put $u = \sin x$ so that $du = \cos x\,dx$ to get

$$\begin{aligned}&= \int u^5(1-u^2)\,du \\ &= \int u^5 - u^7\,du \\ &= \frac{u^6}{6} - \frac{u^8}{8} + C \\ &= \frac{\sin^6 x}{6} - \frac{\sin^8}{8} + C\end{aligned}$$

7. Put $u = x^2 + 1$ so $du = 2x\,dx$. Then

$$\begin{aligned}&\int x^3\sqrt{x^2+1}\,dx \\ &= \int (u-1)u^{1/2} \cdot \frac{1}{2}\,du \\ &= \frac{1}{2}\int u^{3/2} - u^{1/2}\,du \\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \\ &= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C\end{aligned}$$

8. Partial fractions:

$$\frac{5x^2+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Find a common denominator and compare the numerators

$$\begin{aligned} 5x^2+2 &= Ax^2 + A + Bx^2 + Cx \\ &= (A+B)x^2 + Cx + A \end{aligned}$$

Matching coefficients, $A = 2$ and $C = 0$ so $B = 3$. Therefore

$$\begin{aligned} \int \frac{5x^2+2}{x^3+x}\,dx &= \int \left(\frac{2}{x} + \frac{3x}{x^2+1}\right)\,dx \\ &= 2\ln|x| + \frac{3}{2}\ln|x^2+1| + C. \end{aligned}$$

9. First complete the square

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2+2x+10}} &= \int \frac{dx}{\sqrt{x^2+2x+1+9}} \\ &= \int \frac{dx}{\sqrt{(x+1)^2+9}} \end{aligned}$$

Substitute $u = x+1$, $du = dx$,

$$= \int \frac{du}{\sqrt{u^2+0}}$$

Now do a trigonometric substitution, with $u = 3\tan\theta$, so $du = 3\sec^2\theta\,d\theta$ and $\sqrt{u^2+9} = 3\sec\theta$. We then have

$$\begin{aligned} &= \int \frac{3\sec^2\theta d,\theta}{3\sec\theta} \\ &= \int \sec\theta d\,\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{\sqrt{(x+1)^2+9}}{3} + \frac{x+1}{3}\right| + C \end{aligned}$$

10. Put $w = x^2$, so $dw = 2x\,dx$. Then

$$\int x^3\sin(x^2)\,dx = \int \frac{1}{2}w\sin w\,dw$$

Now integration by parts, with $u = x$, $dv = \sin w\,dw$ so that $du = dw$ and $v = -\cos w$, to get

$$\begin{aligned} &= \frac{1}{2}\left(-w\cos w + \int \cos w\,dw\right) \\ &= \frac{1}{2}(-w\cos w + \sin w) + C \\ &= \frac{1}{2}(-x^2\cos x^2 + \sin x^2) + C \end{aligned}$$