Math 2050 Test 3. Fall 2009

1 Use the method of *Integration by Parts* to compute the integral

$$\int x^2 e^{3x}\, dx.$$

2 Use the method of *Trigonometric Substitution* to compute the integral

$$\int \frac{1}{\sqrt{9-x^2}}\, dx.$$

3 Use the method of *Partial Fractions* to compute the integral

$$\int \frac{-5x^2+2x+4}{x^3+2x^2}\, dx.$$

4-8. Evaluate the integrals.

4

$$\int \sin^4 x \cos^3 x\, dx =$$

5

$$\int \frac{3x^2+2}{x^3+2x+8}\, dx =$$

6

$$\int e^x \sin(5x)\, dx =$$

7

$$\int \frac{1}{x^2+4x+8}\, dx =$$

8

$$\int \frac{x^2}{(x-1)^2(x+1)}\, dx =$$

9 Evaluate the improper integral (or explain why the integral diverges)

$$\int_2^\infty \frac{1}{x \ln x}\, dx.$$

10 Evaluate the improper integral (or explain why the integral diverges)

$$\int_2^3 \frac{1}{\sqrt[3]{3x-6}}\, dx.$$

1 Use integration by parts with

$$u = x^2 \qquad dv = e^{3x}\,dx$$
$$du = 2x\,dx \qquad v = \frac{1}{3}e^{3x}$$

to get

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}\,dx.$$

Use integration by parts again with

$$u = x \qquad dv = e^{3x}\,dx$$
$$du = dx \qquad v = \frac{1}{3}e^{3x}$$

to get

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[\frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx\right]$$
$$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right] + C$$
$$= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$$

2 Use a trigonometric substitution, with

$$x = 3\sin\theta$$
$$dx = 3\cos\theta\,d\theta$$
$$\sqrt{9-x^2} = 3\cos\theta$$

to get

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{3\cos\theta}{3\cos\theta}\,d\theta$$
$$= \int 1\,d\theta$$
$$= \theta + C$$
$$= \sin^{-1}(x/3) + C$$

3 First we determine the partial fraction decomposition

$$\frac{-5x^2+2x+4}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

so

$$-5x^2+2x+4$$
$$= Ax(x+2) + B(x+2) + Cx^2.$$

Plugging in $x = 0$,

$$2B = 4 \implies B = 2.$$

Plugging in $x = -2$,

$$4C = -20 \implies C = -5.$$

And plugging in $x = 1$,

$$1 = 3A + 6 - 5 \implies A = 0.$$

Hence,

$$\int \frac{-5x^2+2x+4}{x^3+2x^2}\,dx$$
$$= \int \left(\frac{2}{x^2} - \frac{5}{x+2}\right)\,dx$$
$$= -\frac{2}{x} - 5\ln|x+2| + C$$

4

$$\int \sin^4 x\cos^3 x\,dx$$
$$= \int \sin^4 x(1-\sin^2 x)\cos x\,dx.$$

Now let $u = \sin x$ so $du = \cos x\, dx$ and so

$$\begin{aligned} &= \int u^4(1-u^2)\, du \\ &= \int (u^4 - u^6)\, du \\ &= \frac{1}{5}u^5 - \frac{1}{7}u^7 + C \\ &= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C. \end{aligned}$$

5 Use a substitution, with $u = x^3 + 2x + 8$, $du = (3x^2 + 2)\, dx$, so

$$\begin{aligned} \int \frac{3x^2+2}{x^3+2x+8}\, dx &= \int \frac{1}{u}\, du \\ &= \ln|u| + C \\ &= \ln|x^3 + 2x + 8| + C \end{aligned}$$

6 Let

$$I = \int e^x \sin(5x)\, dx.$$

Integration by parts

$u = \sin(5x)$	$dv = e^x\, dx$
$du = 5\cos(5x)\, dx$	$v = e^x$

so

$$I = e^x \sin(5x) - 5\int \cos(5x) e^x\, dx.$$

And again, with

$u = \cos(5x)$	$dv = e^x\, dx$
$du = -5\sin(5x)\, dx$	$v = e^x$

so

$$\begin{aligned} I =& e^x \sin(5x) \\ &- 5\left[e^x\cos(5x) + \int 5\sin(5x)e^x\, dx\right] \\ =& e^x\sin(5x) - 5e^x\cos(5x) - 25I \end{aligned}$$

Gathering all the I terms on one side,

$$26I = e^x\sin(5x) - 5e^x\cos(5x) + C$$

so

$$I = \frac{1}{26}\left(e^x\sin(5x) - 5e^x\cos(5x)\right) + C.$$

7 First, complete the square,

$$\int \frac{1}{x^2+4x+8}\, dx = \int \frac{1}{(x+2)^2+4}\, dx.$$

Now let $u = x + 2$, so $du = dx$, to get

$$= \frac{du}{u^2+4},$$

and then a trigonometric substitution,

$$\begin{aligned} &u = 2\tan\theta \\ &du = 2\sec^2\theta\, d\theta \\ &\sqrt{u^2+4} = 2\sec\theta \end{aligned}$$

which leads to

$$\begin{aligned} &= \int \frac{2\sec^2\theta}{4\sec^2\theta}\, d\theta \\ &= \int \frac{1}{2}\, d\theta \\ &= \frac{1}{2}\theta + C \\ &= \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C. \end{aligned}$$

8 Partial fractions

$$\begin{aligned} &\frac{x^2}{(x-1)^2(x+1)} \\ &\quad = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \end{aligned}$$

Comparing numerators

$$x^2 = A(x-1)(x+1) + B(x+1) + C(x-1)^2.$$

Plug in $x = 1$ to get

$$2B = 1 \implies B = 1/2.$$

Plug in $x = -1$ to get

$$4C = 1 \implies C = 1/4.$$

Finally, plug in $x = 0$ to get

$$0 = -A + \frac{1}{2} + \frac{1}{4} \implies A = \frac{3}{4}.$$

Therefore

$$\int \frac{x^2}{(x-1)^2(x+1)}\,dx$$

$$= \int \frac{3/4}{x-1}\,dx + \int \frac{1/2}{(x-1)^2}\,dx + \int \frac{1/4}{x+1}\,dx$$

$$= \frac{3}{4}\ln|x-1| - \frac{1}{2}(x-1)^{-1} + \frac{1}{4}\ln|x+1| + C.$$

9

$$\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}\,dx$$

Now do a substitution, $u = \ln x$ so $du = dx/x$.

$$= \lim_{b\to\infty}\int_{x=2}^b \frac{1}{u}\,du$$

$$= \lim_{b\to\infty} \ln|u|\Big|_{x=2}^b$$

$$= \lim_{b\to\infty} \ln|\ln x|\Big|_x^b$$

$$= \lim_{b\to\infty} \ln|\ln b| - \ln|\ln 2|$$

As b approaches infinity, the term $\ln|\ln b|$ approaches infinity as well, so this integral diverges.

10

$$\int_2^3 \frac{1}{\sqrt[3]{3x-6}}\,dx$$

$$= \lim_{a\to 2^+}\int_a^3 (3x-6)^{-1/3}\,dx$$

$$= \lim_{a\to 2^+} \frac{3}{2}(3x-6)^{2/3}\cdot\frac{1}{3}\Big|_a^3$$

$$= \lim_{a\to 2^+}\left[\frac{1}{2}(3)^{2/3} - \frac{1}{2}(3a-6)^{2/3}\right]$$

$$= \frac{3^{2/3}}{2}$$