Math 306 Test 1. Harvey Fall 2008

1 (8 points) Find the limit of the sequence

$$\left\{\frac{5n^2+2n}{3n^2+1}\right\}_{n=1}^{\infty}.$$

2 (6 points) The first few terms in a sequence are given below

$$a_0 = 1 \quad a_1 = -\frac{1}{3} \quad a_2 = \frac{1}{5} \quad a_3 = -\frac{1}{7} \quad a_4 = \frac{1}{9}$$

Based on this pattern, give a formula for a_n, the n^{th} term in the sequence, in terms of n.

3 (6 points) List the first five terms in the sequence

$$\left\{\frac{(n-1)!}{n!}\right\}_{n=1}^{\infty}.$$

4 (8 points) Evaluate the geometric series

$$\sum_{n=2}^{\infty}\frac{3^{n-1}}{4^{n-1}}.$$

5 (8 points) Evaluate the following series

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}.$$

6 (10 points) Determine whether the following series converges conditionally, absolutely, or diverges.

$$\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}.$$

7-13 (9 points each) Do six of the remaining seven problems. Indicate which one you do *not* want me to grade by placing an X in the box. In each, determine whether the series converges or diverges.

☐ **7**

$$\sum_{n=1}^{\infty} e^{1/n}$$

☐ **8**

$$\sum_{n=1}^{\infty}\frac{5n^2-1}{n^3+2n}$$

☐ **9**

$$\sum_{n=1}^{\infty} ne^{-n}$$

☐ **10**

$$\sum_{n=1}^{\infty}\frac{3^n\cdot n!}{(2n)!}$$

☐ **11**

$$\sum_{n=1}^{\infty}\frac{4^n}{n^2\cdot 3^n}$$

☐ **12**

$$\sum_{n=1}^{\infty}\frac{\sqrt{n-1}}{n^2}$$

☐ **13**

$$\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^{n^2}\cdot\left(\frac{1}{2}\right)^n$$

1

$$\lim_{n\to\infty} \frac{5n^2+2n}{3n^2+1} = \lim_{n\to\infty} \frac{5+\frac{2}{n}}{3+\frac{1}{n^2}} = \frac{5}{3}$$

2

$$a_n = \frac{(-1)^n}{2n+1}$$

3

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1/2 \\ a_3 &= 1/3 \\ a_4 &= 1/4 \\ a_5 &= 1/5 \end{aligned}$$

4

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{3^{n-1}}{4^{n-1}} &= \sum_{n=0}^{\infty} \frac{3^{n+1}}{4^{n+1}} \\ &= \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \\ &= \frac{3}{4} \left(\frac{1}{1-3/4}\right) \\ &= \frac{3}{4} \cdot 4 = 3 \end{aligned}$$

5 This is a telescoping series. Use partial fractions

$$\begin{aligned} \frac{1}{(n+1)(n+2)} &= \frac{A}{n+1} + \frac{B}{n+2} \\ &= \frac{A(n+2)+B(n+1)}{(n+1)(n+2)} \end{aligned}$$

Plug into the numerators: $n=-1$ to find $A = 1$ and $n=-2$ to find $B=-1$. Thus

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots \\ &= 1/2 \end{aligned}$$

6 Begin with the Alternating Series Test. To show that the terms are decreasing, look at

$$f(x) = \ln x/x.$$

The derivative is

$$f'(x) = \frac{1-\ln x}{x}$$

which is less than zero when $x > e$. Therefore $f(x)$ is decreasing when $x > e$. And the terms approach zero as n approaches infinity since

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Therefore the series converges.

Now note that for $n \geq 3$,

$$\frac{\ln n}{n} > \frac{1}{n}.$$

The series $\sum 1/n$ diverges (by the p-test, with $p=1$), and so by the Basic Comparison Test, the series $\sum \ln n/n$ must diverge as well. Therefore this series is not absolutely convergent. It is conditionally convergent.

7 Look at the limit of the terms in this series:

$$\lim_{n\to\infty} e^{1/n} = e^0 = 1 \neq 0$$

Therefore, this series diverges by the Divergence Test.

8 Do a limit comparison with $\sum 1/n$.

$$\begin{aligned} \lim_{n\to\infty} \frac{5n^2-1}{n^3+2n} \cdot \frac{n}{1} &= \lim_{n\to\infty} \frac{5n^3-n}{n^3+2n} \\ &= \lim_{n\to\infty} \frac{5-(1/n^2)}{1+(2/n^2)} = 5 \end{aligned}$$

Since $\sum 1/n$ diverges (by the p-test, with $p = 1$), so does this series.

9 To use the Integral Test, we must evaluate the improper integral:

$$\int_1^{\infty} xe^{-x}\, dx$$

This requires integration by parts, with $u = x$, $du = dx$, $dv = e^{-x}dx$, and $v = -e^{-x}$. Then

$$\begin{aligned}
\int_1^\infty xe^{-x}\,dx &= \lim_{a\to\infty}\int_1^a xe^{-x}\,dx \\
&= \lim_{a\to\infty}\left(-xe^{-x}\Big|_1^a + \int_1^a e^{-x}\,dx\right) \\
&= \lim_{a\to\infty}\left(-xe^{-x}\Big|_1^a - e^{-x}\Big|_1^a\right) \\
&= \lim_{a\to\infty}\left(\frac{-x-1}{e^x}\Big|_1^a\right) \\
&= \lim_{a\to\infty}\left(\frac{-a-1}{e^a}+1\right) \\
&\overset{*}{=} \lim_{a\to\infty}\left(\frac{-1}{e^a}+1\right) \\
&= 1
\end{aligned}$$

Because the integral converges, the series will converge as well.

10 Use the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty}\frac{3^{n+1}(n+1)!}{(2(n+1))!}\cdot\frac{(2n)!}{3^n n!} \\
&= \lim_{n\to\infty}\frac{3(n+1)}{(2n+2)(2n+1)} \\
&= 0 < 1
\end{aligned}$$

By the Ratio Test, the series converges.

11 Use the Root Test.

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{4^n}{n^2\cdot 3^n}\right)^{1/n} &= \lim_{n\to\infty}\frac{4}{n^{2/n}\cdot 3} \\
&= 4/3 > 1
\end{aligned}$$

The series diverges by the Root Test.

12 Use the Basic Comparison Test.

$$\frac{\sqrt{n-1}}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

The series $\sum 1/n^{3/2}$ converges by the p-test (with $p = 3/2$), so this series must converge also.

13 Use the Root Test.

$$\begin{aligned}
&\lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^{n^2}\cdot\left(\frac{1}{2}\right)^n\right]^{1/n} \\
&= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{1}{2} \\
&= e\cdot\frac{1}{2} > 1
\end{aligned}$$

This series diverges by the root test.