Math 306 Test 1. Fall 2010

1 (6 points) Is the following sequence increasing, decreasing, or neither? [Give a complete argument– don't just look at the first few terms.]

$$\left\{\frac{n^2}{n^2-1}\right\}_{n=2}^{\infty}$$

2 (8 points) Find the limit of each of the following sequences.

$$(a) \quad \left\{\frac{\sqrt{4n^2+1}}{n}\right\}_{n=1}^{\infty}$$

$$(b) \quad \left\{e^{-n}\right\}_{n=1}^{\infty}$$

3 (6 points) Evaluate the following (convergent) series

$$\sum_{n=3}^{\infty} \frac{2^{n-1}}{3^{n-2}} =$$

4 (8 points) Determine whether the following series converges conditionally, converges absolutely, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

5-13 (9 points each) Do eight of the remaining nine problems. Indicate which one you do *not* want me to grade by placing an $\times$ in the box. For each, determine whether the series converges or diverges. Show your work and explain your reasoning.

5

$$\sum_{n=1}^{\infty} \frac{1}{n^{5/3}}$$

6

$$\sum_{n=1}^{\infty} \frac{n}{3n+5\sqrt{n}}$$

7

$$\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{(2n)!}$$

8

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

9

$$\sum_{n=4}^{\infty} \frac{\sqrt{n+1}}{n^2-3n}$$

10

$$\sum_{n=1}^{\infty} \frac{5^n \cdot n^n}{n!}$$

11

$$\sum_{n=1}^{\infty} \frac{n}{n^3-n^2+1}$$

12

$$\sum_{n=1}^{\infty} \frac{3^n}{n^5 \cdot 2^n}$$

13

$$\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2(1/n)}$$

1 Look at the function

$$f(x) = \frac{x^2}{x^2-1}.$$

Its derivative is

$$\begin{aligned} f'(x) &= \frac{(x^2-1)\cdot 2x - x^2(2x)}{(x^2-1)^2} \\ &= \frac{2x^3-2x-2x^3}{(x^2-1)^2} \\ &= \frac{-2x}{(x^2-1)^2} < 0. \end{aligned}$$

That means that $f(x)$ is decreasing, so the sequence must be decreasing as well.

2

(a)

$$\begin{aligned} \lim_{n\to\infty} \frac{\sqrt{4n^2+1}}{n} &= \lim_{n\to\infty} \sqrt{\frac{4n^2+1}{n^2}} \\ &= \lim_{n\to\infty} \sqrt{4+\frac{1}{n^2}} \\ &= \sqrt{4} = 2. \end{aligned}$$

(b)

$$\lim_{n\to\infty} e^{-n} = \lim_{n\to\infty} \frac{1}{e^n} = 0.$$

3

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{2^{n-1}}{3^{n-2}} &= \sum_{n=0}^{\infty} \frac{2^{n+2}}{3^{n+1}} \\ &= \frac{4}{3}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n \\ &= \frac{4}{3}\cdot\frac{1}{1-(2/3)} \\ &= \frac{4}{3}\cdot 3 \\ &= 4. \end{aligned}$$

4 Use the Alternating Series Test.

(i)

$$\lim_{x\to\infty} \frac{\ln(x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

(ii)

$$f(x) = \frac{\ln x}{x} \implies f'(x) = \frac{x(1/x) - \ln x}{x^2}.$$

This is less than zero for $x > e$, so the sequence $\ln n/n$ is decreasing (for $n > 3$). Therefore the series converges. On the question of absolute convergence, however, note that

$$\frac{\ln(n)}{n} > \frac{1}{n}$$

and $\sum \frac{1}{n}$ diverges (it is the harmonic series). Therefore, by the basic comparison test, $\sum \frac{\ln n}{n}$ diverges. The series given in the problem converges conditionally.

5 This converges: p-test where $p = 5/3 > 1$.

6 Use the divergence test.

$$\lim_{n\to\infty} \frac{n}{3n+5\sqrt{n}} = \lim_{n\to\infty} \frac{1}{3+(5/\sqrt{n})} = \frac{1}{3} \neq 0.$$

Since the terms do not approach zero, the series diverges.

7 Use the ratio test.

$$\begin{aligned} &\lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(2n+2)!}\cdot\frac{(2n)!}{2^n\cdot n!} \\ &= \lim_{n\to\infty} \frac{2(n+1)}{(2n+2)(2n+1)} \\ &= 0 < 1. \end{aligned}$$

The series converges.

8 Use the root test (I had the integral test in mind when I put this problem on the test, but the root test is easier).

$$\lim_{n\to\infty} \left(\frac{n}{e^n}\right)^{1/n} = \frac{1}{e} < 1.$$

The series converges.

9 Limit comparison with $\sum 1/n^{3/2}$.

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{n^2-3n} \cdot \frac{\sqrt{n^3}}{1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^4+n^3}}{n^2-3n} \cdot \frac{1/n^2}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+(1/n)}}{1-(3/n)} = 1.$$

The two series have the same behavior. Since $\sum 1/n^{3/2}$ converges (p-test with $p = 3/2$), the series of this problem must also converge.

10 Use the ratio test

$$\lim_{n\to\infty} \frac{5^{n+1}(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n \cdot n^n}$$

$$= \lim_{n\to\infty} \frac{5(n+1)(n+1)^n}{(n+1)n^n}$$

$$= \lim_{n\to\infty} 5 \cdot \left(\frac{n+1}{n}\right)^n$$

$$= 5e > 1.$$

The series diverges.

11 Limit comparison with $\sum 1/n^2$.

$$\lim_{n\to\infty} \frac{n}{n^3-n^2+1} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n^3}{n^3-n^2+1}$$

$$= \lim_{n\to\infty} \frac{1}{1-(1/n)-(1/n^3)}$$

$$= 1.$$

The two series have the same behavior. Since $\sum 1/n^2$ converges (p-test, with $p = 2$), the series in this problem must also converge.

12 Use the root test.

$$\lim_{n\to\infty} \left(\frac{3^n}{n^5 \cdot 2^n}\right)^{1/n} = \lim_{n\to\infty} \frac{3}{n^{5/n} \cdot 2} = \frac{3}{2} > 1.$$

The series diverges.

13 Limit comparison with $\sum 1/n$.

$$\lim_{n\to\infty} \frac{1}{n + n\cos^2(1/n)} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\cos^2(1/n)} = \frac{1}{2}.$$

Since $\sum 1/n$ diverges, so does the series given in this problem.