Math 3060 Test 1. Fall 2012

1. Is the following sequence increasing, decreasing, or neither? [Give a complete argument–don't just look at the first few terms.]

$$\left\{\frac{\ln(n+3)}{n+3}\right\}_{n=1}^{\infty}$$

2. Compute the limits of the following sequences (if they exist).

$$(a)\quad \left\{\frac{n^2+2n+3}{5n^2+n-1}\right\} \qquad (b)\ \left\{e^{n/(2n-1)}\right\}.$$

3. Evaluate the following (convergent) series

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{2n+1}}.$$

For problems 4–10, determine whether the series converges or diverges. Explain your reasoning. Each problem is worth 8 points.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\cdot 3^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n!3^n}{(2n)!}$

7. $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{\sqrt{n}}$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{n^2+4}$

9. $\displaystyle\sum_{n=0}^{\infty} \sqrt{\frac{n+1}{4n+5}}$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{2n+5}{\sqrt[3]{n^8+1}}$

11. Does the following series converge absolutely, converge conditionally, or diverge?

$$\sum (-1)^n \frac{2^n}{n!}$$

SOLUTIONS

1. Look at the function $f(x) = \ln(x+3)/(x+3)$. Then

$$f'(x) = \frac{(x+3)\cdot\dfrac{1}{x+3} - \ln(x+3)\cdot 1}{(x+3)^2}$$

$$= \frac{1-\ln(x+3)}{(x+3)^2}.$$

When $x > 0$, $f'(x) < 0$, so the function, and therefore the sequence, is decreasing.

2.

$$(a) \quad \lim_{n\to\infty} \frac{n^2+2n+3}{5n^2+n-1}$$

$$= \lim_{n\to\infty} \frac{1+(2/n)+(3/n^2)}{5+(1/n)-(1/n^2)}$$

$$= 1/5$$

$$(b) \quad \lim_{n\to\infty} e^{n/(2n-1)}$$

$$= \lim_{n\to\infty} e^{1/[2-(1/n)]}$$

$$= e^{1/2}$$

3.

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{2n+1}} = \sum_{n=0}^{\infty} \frac{3^n}{4^{2n+3}}$$

$$= \frac{1}{64}\sum_{n=0}^{\infty}\left(\frac{3}{16}\right)^n$$

$$= \frac{1}{64}\cdot\frac{1}{1-(3/16)}$$

$$= \frac{1}{64}\cdot\frac{16}{13}$$

$$= \frac{1}{52}$$

4. Use the root test–

$$\lim_{n\to\infty}\left|\frac{n}{e^{n^2}}\right|^{1/n} = \lim_{n\to\infty}\frac{n^{1/n}}{e^n} = 0 < 1.$$

The series converges .

5. Use the root test–

$$\lim_{n\to\infty}\left|\frac{1}{n^{1/2}\cdot 3^n}\right|^{1/n} = \lim_{n\to\infty}\frac{1}{n^{1/2n}\cdot 3} = \frac{1}{3} < 1.$$

The series converges.

6. Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(n+)!3^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n!3^n}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)3}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty}\frac{3}{2(2n+1)} = 0 < 1.$$

The series converges.

7. Use the comparison test:

$$\frac{\ln n}{\sqrt{n}} > \frac{1}{\sqrt{n}}$$

The series $\sum \frac{1}{\sqrt{n}}$ diverges (p-test with $p = 1/2$). By comparison, then $\sum \frac{\ln n}{\sqrt{n}}$ diverges as well.

8. Use the Alternating Series Test. (1) The limit does approach zero:

$$\lim_{n\to\infty}\frac{n}{n^2+4} = 0.$$

(2) Look at the function $f(x) = x/(x^2+4)$:

$$f'(x) = \frac{(x^2+4)-x(2x)}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}.$$

For $x > 2$, $f' < 0$, and so $f(x)$ (and hence the sequence) is decreasing. Therefore the series converges.

9. Use the Divergence Test.

$$\lim_{n\to\infty}\sqrt{\frac{n+1}{4n+5}} = \lim_{n\to\infty}\sqrt{\frac{1+(1/n)}{4+(5/n)}} = \frac{1}{2} \neq 0.$$

The series diverges.

10. Note that

$$\frac{n}{\sqrt[3]{n^8}} = \frac{1}{n^{5/3}},$$

so do a limit comparison with the series $\sum \frac{1}{n^{5/3}}$.

$$\lim_{n\to\infty} \frac{2n+5}{\sqrt[3]{n^8+1}} \cdot \frac{n^{5/3}}{1}$$

$$= \lim_{n\to\infty} \frac{2n+5}{\sqrt[3]{n^8+1}} \cdot \frac{n^{8/3}}{n}$$

$$= \lim_{n\to\infty} \left(\frac{2n+5}{n}\right) \cdot \sqrt[3]{\frac{n^8}{n^8+1}}$$

$$= \lim_{n\to\infty} \left(2 + \frac{5}{n}\right) \cdot \sqrt[3]{\frac{1}{1+(1/n^8)}}$$

$$= 2.$$

The comparison is valid. Since $\sum \frac{1}{n^{5/3}}$ converges (p-test, $p = 5/3$), so does the series $\sum \frac{2n+5}{\sqrt[3]{n^8+1}}$.

11. Use the ratio test:

$$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1.$$

The series converges absolutely.