

MATH 306 TEST 1. SPRING 2009

No books, notes, or calculators are allowed. Please show all of your work.

1 (8 points) Find the limit of the sequence

$$\left\{ \frac{3n+1}{5n-2} \right\}_{n=1}^{\infty}.$$

2 (6 points) The first few terms in a sequence are given below

$$a_0 = 1 \quad a_1 = -\frac{1}{2} \quad a_2 = \frac{1}{4} \quad a_3 = -\frac{1}{8} \quad a_4 = \frac{1}{16}$$

Based on this pattern, give a formula for a_n , the n^{th} term in the sequence, in terms of n .

3 (6 points) Is the following sequence increasing, decreasing, or neither?

$$\left\{ \frac{n^2+2}{n^2+1} \right\}_{n=1}^{\infty}$$

4 (8 points) Find the sum of the following (convergent) series

$$\sum_{n=2}^{\infty} \frac{3^{n-1}}{4^{n-1}}.$$

5 (8 points) Find the sum of the following (convergent) series

$$\sum_{n=1}^{\infty} \frac{4}{(n+1)(n+3)}.$$

6 (10 points) Determine whether the following series converges conditionally, absolutely, or diverges.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^3+4}.$$

7-13 (9 points each) Do six of the remaining seven problems. Indicate which one you do *not* want me to grade by placing an X in the box. In each, determine whether the series converges or diverges.

7

$$\sum_{n=1}^{\infty} \frac{5n}{n^5-3}$$

8

$$\sum_{n=1}^{\infty} \frac{2+4^n}{3+6^n}$$

9

$$\sum_{n=2}^{\infty} \frac{5}{n(\ln n)^2}$$

10

$$\sum_{n=1}^{\infty} \frac{5^n}{n^4 2^n}$$

11

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{2n}\sqrt{2n+1}}$$

12

$$\sum_{k=1}^{\infty} \frac{2^k \cdot k!}{k^k}$$

13

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

1

$$\lim_{n \rightarrow \infty} \frac{3n+1}{5n-2} = \lim_{n \rightarrow \infty} \frac{3+1/n}{5-2/n} = 3/5$$

2

$$a_n = (-1/2)^n$$

3 Consider the function

$$f(x) = \frac{x^2+2}{x^2+1}$$

Compute its derivative

$$\begin{aligned} f'(x) &= \frac{(x^2+1)(2x) - (x^2+2)(2x)}{(x^2+1)^2} \\ &= \frac{-2x}{(x^2+1)^2} < 0 \end{aligned}$$

Therefore the function (and hence the sequence) is decreasing.

4

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{3^{n+1}}{4^{n+1}} = \frac{3}{4} \sum_{n=0}^{\infty} (3/4)^n \\ &= \frac{3}{4} \cdot \frac{1}{1-3/4} = 3 \end{aligned}$$

5 Partial fractions decomposition:

$$\begin{aligned} \frac{4}{(n+1)(n+3)} &= \frac{A}{n+1} + \frac{B}{n+3} \\ 4 &= A(n+3) + B(n+1) \end{aligned}$$

Plug in $n = -3$ to find that $B = -2$. Plug in $n = -1$ to find that $A = 2$.

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{2}{n+1} - \frac{2}{n+3} \\ &= \frac{2}{2} - \frac{2}{4} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{5} + \dots \\ &= 1 + \frac{2}{3} = 5/3 \end{aligned}$$

6 Look at the series $\sum n/(n^3+4)$. Do a limit comparison with $\sum 1/n^2$:

$$\lim_{n \rightarrow \infty} \frac{n}{n^3+4} \cdot \frac{n^2}{1} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3+4} = 1$$

Therefore the two series have the same behavior. Since $\sum 1/n^2$ converges, so does $\sum n/(n^3+4)$. Therefore the original series converges absolutely.7 Do a limit comparison with $\sum 1/n^4$:

$$\lim_{n \rightarrow \infty} \frac{5n}{n^5-3} \cdot \frac{n^4}{1} = \lim_{n \rightarrow \infty} \frac{5n^5}{n^5-3} = 5.$$

The comparison is valid. Since $\sum 1/n^4$ converges (p -test, with $p = 4$), this series also converges.8 Limit comparison with $\sum (2/3)^n$:

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{2+4^n}{3+6^n} \cdot \frac{3^n}{2^n} \\ &= \lim_{n \rightarrow \infty} \frac{2+4^n}{3+6^n} \cdot \frac{6^n}{4^n} \\ &= \lim_{n \rightarrow \infty} \frac{2/4^n+1}{3/6^n+1} = 1 \end{aligned}$$

Since $\sum (2/3)^n$ converges (geometric series with $x = 2/3$), our series also converges.9 Use the integral test, and the substitution $u = \ln x$, $du = 1/x dx$:

$$\begin{aligned} \int_2^{\infty} \frac{5}{x(\ln x)^2} dx &= \lim_{a \rightarrow \infty} \int_{\ln 2}^{\infty} \frac{5}{u^2} du \\ &= \lim_{a \rightarrow \infty} -5/u \Big|_{x=\ln 2}^{x=a} \\ &= \lim_{a \rightarrow \infty} \left(\frac{-5}{a} + \frac{5}{\ln 2} \right) \\ &= 5/\ln 2 \end{aligned}$$

The integral converges, so the series converges.

10 Use the root test

$$\lim_{n \rightarrow \infty} \left(\frac{5^n}{n^4 2^n} \right)^{1/n} = \lim_{n \rightarrow \infty} \frac{5}{n^{4/n} \cdot 2} = 5/2 > 1$$

Therefore the series diverges.

11 Look at the limit of the terms

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{2n}\sqrt{2n+1}} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{4n^2 + 2n}}$$

Divide through in the top and bottom by n :

$$= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{4 + 2/n}} = 1/4 \neq 0.$$

The series diverges by the diverges test.

12 Use the ratio test:

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{2^{k+1}(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{2^k \cdot k!} \\ &= \lim_{k \rightarrow \infty} \frac{2(k+1)k!}{(k+1)(k+1)^k} \cdot \frac{k^k}{k!} \\ &= \lim_{k \rightarrow \infty} \frac{2k^k}{(k+1)^k} = 2/e < 1 \end{aligned}$$

The series converges.

13 Use the ratio test:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{[(n+1)!]^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2(2n)![n!]^2}{(2n+2)(2n+1)(2n)!(n!)^2} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{4n^2 + 6n + 2} = \frac{1}{4} < 1 \end{aligned}$$

The series converges.