Math 306 Test 2. Harvey Fall 2008

1-2 (10 points each) Find the interval of convergence of the power series. Be sure to test the endpoints, where applicable.

1

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n \cdot n^{3/2}}$$

2

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$$

3 (10 points) Compute the Taylor series for $f(x) = \cos(x)$ expanded about the point $c = \pi$ using the "definition"

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

4 (8 points) Continuing with the function and the Taylor series calculated in the previous problem, now calculate the remainder

$$|R_{n+1}(x)| \leq \frac{M}{(n+1)!} |x-c|^{n+1}$$

and show that the series converges to the function for all x.

5-7 (8 points each) Modify known Taylor series to find the Taylor series (expanded about 0) for each of the following functions

5 $\quad f(x) = x^5 \sin(3x^2)$

6 $\quad f(x) = e^{2x-1}$

7 $\quad f(x) = \ln(2+4x)$

8-9 (8 points each) Working from the geometric series, find Taylor series (expanded about 0) for each of the following functions. What is the radius of convergence?

8 $\quad f(x) = \dfrac{1}{2+x}$

9 $\quad f(x) = \dfrac{1}{(1-3x)^2}$

10 (8 points) Use a series representation of $\sin x$ to compute the limit

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}.$$

11 (8 points) Find a series solution to the integral

$$\int \frac{\cos x - 1}{x^2}\, dx.$$

12 (10 points) Use a series to approximate the value of $\ln(1.5)$ to within a margin of error of $1/50$.

1 Use the root test

$$\lim_{n\to\infty}\left(\frac{|x-1|^n}{2^n n^{3/2}}\right)^{1/n} = \lim_{n\to\infty}\frac{|x-1|}{2n^{3/2n}} = \frac{|x-1|}{2}$$

In order for this to be less than one, $|x-1|<2$, so $-1<x<3$. Testing the endpoints, when $x=3$, the series $\sum 1/n^{3/2}$ is absolutely convergent, so the series converges at both endpoints. The interval of convergence is $[-1,3]$.

2 Use the ratio test

$$\lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}|x|^{n+1}\cdot\frac{n!}{2^n|x|^n} = \lim_{n\to\infty}\frac{2|x|}{n+1}=0$$

The series converges for all x, so the interval of convergence is $(-\infty,\infty)$.

3 Compute derivatives

n	$f^n(x)$	$f^n(\pi)$
0	$\cos x$	-1
1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
5	$-\sin x$	0
6	$-\cos x$	1

The Taylor series is

$$\cos x = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k)!}(x-\pi)^{2k}$$

4 For all n, $|f^{(n)}|\le 1$ so we may use $M=1$ and

$$|R_{n+1}(x)|\le\frac{1}{(n+1)!}|x-\pi|^{n+1}$$

Using the ratio test, we can see that the sum $\sum|x-\pi|^{n+1}/(n+1)!$ converges for all x. Therefore the individual terms must go to zero, so

$$\lim_{n\to\infty}\frac{|x-\pi|^{n+1}}{(n+1)!}=0$$

for all x. The remainder goes to zero for all x, so the Taylor series converges to the function for all x.

5

$$f(x) = x^5\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(3x^2)^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}3^{2n+1}x^{4n+7}$$

6

$$f(x)=\frac{1}{e}e^{2n} = \frac{1}{e}\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n\cdot x^n}{e\cdot n!}$$

7

$$f(x)=\ln(2)+\ln(1+2x) = \ln(2)+\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{n+1}}{n+1} = \ln(2)+\sum_{n=0}^{\infty}(-1)^n\frac{2^{n+1}x^{n+1}}{n+1}$$

8

$$f(x)=\frac{1}{2}\cdot\frac{1}{1-(-x/2)} = \frac{1}{2}\sum_{n=0}^{\infty}(-x/2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

The radius of convergence is 2.

9

$$\begin{aligned} f(x) &= \frac{d}{dx}\left(\frac{1/3}{1-3x}\right) \\ &= \frac{d}{dx}\sum_{n=0}^{\infty} 3^{n-1}x^n \\ &= \sum_{n=1}^{\infty} 3^{n-1}nx^{n-1} \end{aligned}$$

The radius of convergence is 1/3.

10

$$\begin{aligned} &\lim_{x\to 0}\frac{1}{x^3}\left(\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)-x\right) \\ &= \lim_{x\to 0}\frac{1}{x^3}\left(\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots\right)-x\right) \\ &= \lim_{x\to 0}\frac{1}{x^3}\left(-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots\right) \\ &= \lim_{x\to 0}\left(-\frac{1}{3!}+\frac{1}{5!}x^2-\cdots\right) \\ &= -1/6 \end{aligned}$$

11

$$\begin{aligned} &\int \frac{1}{x^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}-1\right)dx \\ &= \int \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n-2}\,dx \\ &= C+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n)!}x^{2n-1} \end{aligned}$$

12

$$\begin{aligned} \ln(1+x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} \\ &= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \\ \ln(1+.5) &= \frac{1}{2}-\frac{1}{8}+\frac{1}{24}-\frac{1}{64}+\cdots \end{aligned}$$

Since the series is alternating, we can look at the next term to determine the error. Therefore

$$\begin{aligned} \ln(1+.5) &= \frac{1}{2}-\frac{1}{8}+\frac{1}{24} \\ &= 5/12 \end{aligned}$$