Math 3060 Test 2. Fall 2010

1-2 (10 points each) Find the interval of convergence of the following power series. Be sure to test the endpoints.

1

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{2^n \cdot n^2}$$

2

$$\sum_{n=0}^{\infty}\frac{(x-1)^n}{3n+1}$$

3 (10 points) Use the definition of the Taylor series to compute the Taylor series for $\sin x$ expanded about the point $c = \pi$.

4-7 (8 points each) Manipulate a known series to find the Taylor series for each function (expanded about $c = 0$).

4

$$f(x) = x^3 \sin(2x)$$

5

$$f(x) = xe^{-x}$$

6

$$f(x) = \frac{1}{1+3x^2}$$

7

$$f(x) = \frac{1}{(1+x)^2}$$

8 (6 points) What is the exact value of the following series?

$$\sum_{n=0}^{\infty}\frac{(-1)^n \cdot 3^n}{n!}$$

9 (6 points) Use the Taylor series for sine and cosine to evaluate following limit [no credit for using any other method]

$$\lim_{x\to 0}\frac{1-\cos x}{\sin(x^2)}.$$

10 (8 points) Find a series solution to the integral

$$\int x^2 e^{x^2}\,dx$$

11-12 (10 points each) Use Taylor series to estimate the indicated value to within 1/500. Be sure to justify your decision regarding the number of terms required for this level of precision.

11 $\cos(1)$

12 $e^{1/10}$

1 Use the root test:

$$\lim_{n\to\infty}\left|\frac{(-1)^n x^n}{2^n n^2}\right|^{1/n} = \lim_{n\to\infty}\frac{|x|}{2n^{2/n}} = \frac{|x|}{2}.$$

The series converges if $|x| < 2$ and diverges if $|x| > 2$. Now to test the endpoints. When $x = 2$, $\sum(-1)^n/n^2$ converges by the Alternating Series Test. When $x = -2$, $\sum 1/n^2$ converges by the p-test ($p = 2$). Therefore, the interval of convergence is $[-2, 2]$.

2 Use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\frac{|x-1|^{n+1}}{3n+4}\cdot\frac{3n+1}{|x-1|^n} &= \lim_{n\to\infty}\frac{3n+1}{3n+4}\cdot|x-1| \\ &= |x-1|.\end{aligned}$$

This converges for $0 < x < 2$ and diverges if $x < 0$ or $x > 2$. Now to test the endpoints. When $x = 0$, $\sum(-1)^n/(3n+1)$ converges by the alternating series test. When $x = 2$, $\sum 1/(3n+1)$ diverges by a limit comparison with the harmonic series. The interval of convergence is $[0, 2)$.

3

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\sin x$	0
1	$\cos x$	-1
2	$-\sin x$	0
3	$-\cos x$	1
4	$\sin x$	0
5	$\cos x$	-1
$\vdots$	$\vdots$	$\vdots$

The Taylor series is

$$\frac{-1}{1!}(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}$$

4

$$\begin{aligned}f(x) &= x^3\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x)^{2n+1} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}x^{2n+4}\end{aligned}$$

5

$$f(x) = x\cdot\sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{n+1}$$

6

$$\begin{aligned}f(x) &= \frac{1}{1-(-3x^2)} \\ &= \sum_{n=0}^{\infty}(-3x^2)^n \\ &= \sum_{n=0}^{\infty}(-3)^n x^{2n}\end{aligned}$$

7

$$\begin{aligned}f(x) &= \frac{1}{(1+x)^2} = \frac{d}{dx}\left(\frac{-1}{1+x}\right) \\ &= \frac{d}{dx}\left(-\sum_{n=0}^{\infty}(-x)^n\right) \\ &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^{n+1}x^n\right) \\ &= \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}\end{aligned}$$

8

$$\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 3^n}{n!} = e^{-3}$$

9

$$\lim_{n\to\infty}\frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)}{\frac{x^2}{1!}-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\cdots}$$

$$=\lim_{n\to\infty}\frac{\frac{x^2}{2!}-\frac{x^4}{4!}-\cdots}{\frac{x^2}{1!}-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\cdots}$$

$$=\lim_{n\to\infty}\frac{\frac{1}{2}-\frac{x^2}{4!}+\cdots}{1-\frac{x^4}{3!}+\frac{x^8}{5!}-\cdots}$$

$$=1/2$$

10

$$\int x^2e^{x^2}\,dx=\int\left(x^2\sum_{n=0}^{\infty}\frac{1}{n!}(x^2)^n\right)\,dx$$

$$=\int\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^{2n+2}\right)\,dx$$

$$=C+\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{x^{2n+3}}{2n+3}$$

11

$$\cos(1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(1)^{2n}$$

$$=1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\frac{1}{8!}-\cdots$$

Note that $1/6! = 1/720 < 1/500$, so, to within $1/500$,

$$\cos(1)=1-\frac{1}{2}+\frac{1}{24}=\frac{13}{24}.$$

12

$$e^{1/10}=\sum_{n=0}^{\infty}\frac{1}{n!}(1/10)^n$$

$$=1+\frac{1}{10}+\frac{1}{2!}\cdot\frac{1}{100}+\frac{1}{3!}\frac{1}{1000}+\cdots$$

$$R_n(x)=\frac{M}{(n+1)!}(1/10)^{n+1}$$

$$<\frac{2}{(n+1)!}\cdot\frac{1}{10^{n+1}}$$

Note that when $n = 2$,

$$R_2(x)<\frac{2}{3!}\cdot\frac{1}{10^3}=\frac{2}{6000}<\frac{1}{500},$$

so to within $1/500$,

$$e^{1/10}\approx 1+\frac{1}{10}+\frac{1}{200}=\frac{221}{200}.$$