Math 306 Test 2. Spring 2009

No books, notes, or calculators are allowed. Please show all of your work.

1-2 (10 points each) Find the interval of convergence of the power series. Be sure to check the endpoints.

1

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 5^n} x^n$$

2

$$\sum_{n=1}^{\infty} \frac{3^n}{n^3 \cdot 2^n} (x-2)^n$$

3 (10 points) Use the definition of the Taylor series as the sum

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n.$$

to *directly* calculate the Taylor series expanded about the point $c = 0$ for the function $f(x) = \sin(2x)$. Do *not* derive this series from a known series.

4 (6 points) The remainder for a Taylor series can be estimated using the inequality

$$|R_n(x)| = \frac{M}{(n+1)!} |x-c|^{n+1},$$

where M is the maximum value of $f|^{(n+1)}|$ between x and c. Referring back to the function in problem 3, what is the value of M (in terms of n)?

5-9 Use a known series to find the Taylor series expanded about the point $c = 0$ for each of the following functions.

5 (6 points) $\quad f(x) = \dfrac{3}{1+2x^2}$

6 (8 points) $\quad f(x) = \dfrac{1}{(1+2x)^2}$

7 (6 points) $\quad f(x) = e^{-2x^2}$

8 (6 points) $\quad f(x) = \cos(-5x^3)$

9 (6 points) $\quad f(x) = \ln(3 - 6x)$

10 (10 points) Use a series to estimate $e^{-1/3}$ to within an error of $1/100$.

11 (6 points) Determine the *exact* value of the series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\pi^{2n}}{4^n}$$

12 (6 points) Use a series to compute the limit

$$\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^4}$$

13 (10 points) Find a series solution to the integral

$$\int_0^{\pi} \cos(x^2)\, dx$$

SOLUTIONS

1 First use the root test:

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n\cdot 5^n}x^2\right|^{1/n} = |x|/5 < 1$$
$$\implies |x| < 5$$

Now test the endpoints. When $x = -5$,

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

diverges (harmonic series), but when $x = 5$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

converges (alternating series test– the terms are decreasing and approaching zero). Therefore the interval of convergence is $(-5, 5]$.

2 Again start with the root test:

$$\lim_{n\to\infty}\left|\frac{3^n}{n^3 2^n}(x-2)^n\right|^{1/n} = \frac{3}{2}|x-2| < 1$$
$$\implies |x-2| < 2/3$$

This means the distance from x to 2 is less than $2/3$, so x lies in the interval $(4/3, 8/3)$. Now to test the endpoints. When $x = 4/3$,

$$\sum_{n=1}^{\infty}\frac{3^n}{n^3 2^n}(-2/3)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$$

This series converges absolutely (by the p-test, with $p = 3$). Therefore the series converges at both endpoints, and the interval of convergence is $[4/3, 8/3]$.

3

n	$f^{(n)}$	$f^{(n)}(0)$
0	$\sin(2x)$	0
1	$2\cos(2x)$	2
2	$-4\sin(2x)$	0
3	$-8\cos(2x)$	-8
4	$16\sin(2x)$	0
5	$32\cos(2x)$	32

Let $k = 2n+1$ to only get the nonzero terms. The Taylor series is then:

$$\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k+1}}{(2k+1)!}x^{2k+1}$$

4 $M = 2^n$

5

$$f(x) = 3\cdot\sum_{n=0}^{\infty}(-2x^2)^n$$
$$= \sum_{n=0}^{\infty}3(-2)^n x^{2n}$$

6 Note that

$$\int\frac{dx}{(1+2x)^2} = \frac{-1}{2(1+2x)} + C$$

Therefore

$$f(x) = \frac{d}{dx}\left[-\frac{1}{2}\sum_{n=0}^{\infty}(-2x)^n\right]$$
$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-2)^{n-1}x^n\right]$$
$$= \sum_{n=1}^{\infty}(-2)^{n-1}\cdot nx^{n-1}$$

7

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}(-2x^2)^n$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}(-2)^n x^{2n}$$

8

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(-5x^3)^{2n}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}25^n x^{6n}$$

9

$$\begin{aligned}f(x) &= \ln 3 + \ln(1-2x)\\ &= \ln 3 + \sum_{n=0}^{\infty}(-1)^n \frac{(-2x)^{n+1}}{n+1}\\ &= \ln 3 + \sum_{n=0}^{\infty}(-1)^n \frac{(-2)^{n+1}}{n+1}x^{n+1}\end{aligned}$$

Note that this series only converges to the function in the interval $(-1/2, 1/2)$.

10

$$\begin{aligned}e^{-1/3} &= \sum_{n=0}^{\infty}\frac{1}{n!}(-1/3)^n\\ &= 1 - \frac{1}{3} + \frac{1}{2}\cdot\frac{1}{9} - \frac{1}{6}\cdot\frac{1}{27} + \cdots\end{aligned}$$

This is an alternating series, so the next term is a bound on the error. Since the fourth term is less than 1/100, we may approximate:

$$e^{-1/3} \approx 1 - \frac{1}{3} + \frac{1}{18} = \frac{13}{18}$$

11

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\pi/2)^{2n} = \cos(\pi/2) = 0.$$

12

$$\begin{aligned}&= \lim_{x\to 0}\frac{1}{x^4}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2x)^{2n} - 1 + 2x^2\right]\\ &= \lim_{x\to 0}\frac{1}{x^4}\left[1 - \frac{1}{2!}4x^2 + \frac{1}{4!}16x^4 - \cdots - 1 + 2x^2\right]\\ &= \lim_{x\to 0}\left[\frac{16}{24} - \cdots\right]\\ &= 2/3\end{aligned}$$

13

$$\begin{aligned}&= \int_0^{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{4n}\,dx\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\cdot\frac{x^{4n+1}}{4n+1}\Big|_0^{\pi}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\cdot\frac{\pi^{4n+1}}{4n+1}\end{aligned}$$