Math 306 Test 4. Spring 2009

No books, notes, or calculators are allowed. Please show all of your work.

1 (8 points) Sketch the domain of the function $f(x, y) = \sqrt{x - 2y}$.

2 (8 points) Sketch the level curves of the function $f(x, y) = y^2 - x$ corresponding to the values $c = 0$, $c = 2$, and $c = 4$.

3 (8 points) Demonstrate that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x+4y}.$$

4 (10 points) Let $f(x, y) = \ln x + 3xy + e^{2y}$.

(a) Compute the partial derivatives f_x and f_y.

(b) Compute the second derivatives f_{xx}, f_{xy} and f_{yy}.

5 (8 points) Find the gradient vector of the function $f(x, y, z) = \frac{2x}{3+y} + z$ at the point $(1, 1, 1)$.

6 (10 points) Compute the directional derivative of $f(x, y) = xe^y$ at the point $(2, 0)$ in the direction of the vector $\langle -1, 1\rangle$.

7 (10 points) Find the equation of the tangent plane to the surface $z = x^2 + y^2 + 3x - 2y$ at the point $(1, 1, 3)$.

8 (8 points) Suppose that

$$f(x, y) = 2x^2 - y^2$$

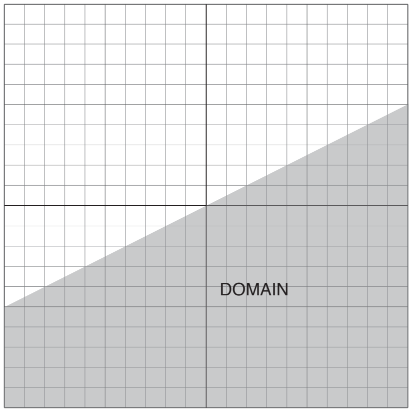
and that $x = r\cos\theta + 1$ and $y = r\sin\theta$. Use the chain rule to compute $\partial f/\partial r$ and $\partial f/\partial\theta$.

9 (8 points) Let $f(x, y) = \cos x \cdot e^y$. Give a vector which indicates the direction of maximum rate of change of this function at the point $(\pi/4, -1)$. What is the rate of change of f in this direction?

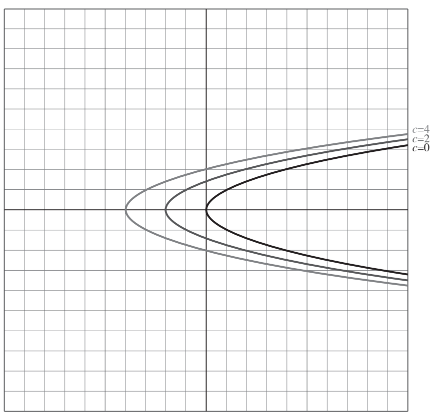
10 (10 points) Find the critical point(s) of the function $f(x, y) = 2x - 2xy + 2y^2$.

11 (10 points) Find the absolute maximum and minimum values of $f(x, y) = x - x^2 + y - y^2$ on the closed rectangle $R = \{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 2\}$.

1



2



3 Along the path $r(t) = \langle t, 0\rangle$, the limit is

$$\lim_{t\to 0} \frac{t+0}{t+4\cdot 0} = 1.$$

Along the path $r(t) = \langle 0, t\rangle$, the limit is

$$\lim_{x\to 0} \frac{0+t}{0+4\cdot t} = \frac{1}{4}.$$

Since the limits along these two paths are not the same, the limit does not exist.

4

$$\begin{aligned} (a)\quad & f_x = 1/x + 3y \\ & f_y = 3x + 2e^{2y} \\ (b)\quad & f_{xx} = -1/x^2 \\ & f_{xy} = 3 \\ & f_{yy} = 4e^{2y} \end{aligned}$$

5

$$\nabla f = \left\langle \frac{2}{3+y}, \frac{-2x}{(3+y)^2}, 1 \right\rangle$$
$$\nabla f(1,1,1) = \langle 1/2, -1/8, 1\rangle$$

6

$$\begin{aligned} \nabla f &= \langle e^y, xe^y\rangle \\ \nabla f(2,0) &= \langle 1, 2\rangle \\ u &= \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle \end{aligned}$$

$$\begin{aligned} f_u &= \nabla f \cdot u \\ &= \langle 1,2\rangle\langle -1/\sqrt{2}, 1/\sqrt{2}\rangle \\ &= 1/\sqrt{2} \end{aligned}$$

7 This is a level surface of the function

$$f(x,y,z) = x^2 + y^2 + 3x - 2y - z$$

The normal vector to this surface is

$$\begin{aligned} \nabla f &= \langle 2x+3, 2y-2, -1\rangle \\ \nabla f(1,1,3) &= \langle 5, 0, -1\rangle \end{aligned}$$

The equation of the tangent plane is

$$\begin{aligned} \langle 5,0,-1\rangle \cdot \langle x-1, y-1, z-3\rangle &= 0 \\ 5(x1) - (z-3) &= 0 \\ 5x - z &= 2 \end{aligned}$$

8

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial y}\partial r \\ &= 4x\cos\theta - 2y\sin\theta \\ &= 4(r\cos\theta + 1)\cos\theta - 2r\sin^2\theta \end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial y}\partial\theta \\ &= 4x(-r\sin\theta) - 2y(r\cos\theta) \\ &= -4(r\cos\theta+1)\cdot r\sin\theta - 2r^2\sin\theta\cos\theta\end{aligned}$$

9

$$\nabla f = \langle 2-2y, -2x+4y\rangle$$

Set $\nabla f = 0$ to find the critical points

$$\begin{cases} 2-2y = 0 \\ -2x+4y = 0 \end{cases} \implies x = 2,\ y = 1$$

The only critical point is $(2, 1)$.

10

$$\nabla f = \langle 1-2x, 1-2y\rangle$$

There is a critical point in the interior:

$$\nabla f = 0: \quad (1/2, 1/2)$$

We have to parametrize the four lines forming the boundary of R.

$$\begin{aligned}&x = 0: \\ &f = y - y^2 \\ &f' = 1-2y \qquad f' = 0: y = 1/2\end{aligned}$$

$$\begin{aligned}&x = 2: \\ &f = -2 + y - y^2 \\ &f' = 1-2y \qquad f' = 0: y = 1/2\end{aligned}$$

$$\begin{aligned}&y = 0 \\ &f = x - x^2 \\ &f' = 1-2x \qquad f' = 0: x - 1/2\end{aligned}$$

$$\begin{aligned}&y = 2 \\ &f = x - x^2 - 2 \\ &f' = 1-2x \qquad f' = 0: x = 1/2\end{aligned}$$

These contribute four more points, $(0, 1/2)$, $(2, 1/2)$, $(1/2, 0)$, and $(1/2, 2)$. In addition, we must test the four corners, $(0, 0)$, $(0, 2)$, $(2, 0)$ and $(2, 2)$.

$$\begin{aligned}&f(1/2, 1/2) = 1/2 \\ &f(0, 1/2) = 1/4 \\ &f(0, 1/2) = 1/4 \\ &f(2, 1/2) = -7/4 \\ &f(1/2, 0) = 1/4 \\ &f(1/2, 2) = -7/4 \\ &f(0, 0) = 0 \\ &f(0, 2) = -2 \\ &f(2, 0) = -2 \\ &f(2, 2) = -4\end{aligned}$$

The absolute maximum is $1/2$ at $(1/2, 1/2)$. The absolute minimum is -4 at $(2, 2)$.