MATH 3060 TEST 4. FALL 2010

1 For the surface $z = x^3 + y$, sketch each of the level curves $z = c$ for the values $c = 0$, 2, and 4.

2 Sketch the graph of the domain of each function

$$(a) \quad f(x, y) = \sqrt{x - y}$$
$$(b) \quad f(x, y) = \ln(x^2 + y) + 3x^2$$

3 If the limit does not exist, explain why. If the limit does exist, compute it.

$$(a) \quad \lim_{(x,y)\to(0,0)} \frac{x^2 + 4y^2}{x^2 + 2y^2}$$

$$(b) \quad \lim_{(x,y)\to(0,0)} e^{x^2 - 2y^2}$$

4 Compute the gradient $\nabla f(x, y, z)$ for the function

$$f(x, y, z) = \ln(2x + 4y) + \sin(z^2).$$

5 Compute the partial derivatives f_x, f_y, f_{xx} and f_{yy} for the function

$$f(x, y) = \frac{x}{y^2} + \sin(xy).$$

6 Compute the directional derivative of $f(x, y) = (2x + y)^{-1}$ at the point $(1, 1)$ in the direction given by the vector $\langle 1, 3\rangle$.

7 Consider the surface

$$z = x^2 - 2xy + 2y^2.$$

Indicate with a unit vector the direction of greatest increase at the point $(1, 2)$.

8 Consider the surface

$$z = x^2 - 2xy + 2y^2.$$

What is the equation of the tangent plane to the surface at the point $(1, 2, 5)$?

9 Find the critical point(s) of the function

$$f(x, y) = x^2 + y^2 - xy - 7y.$$

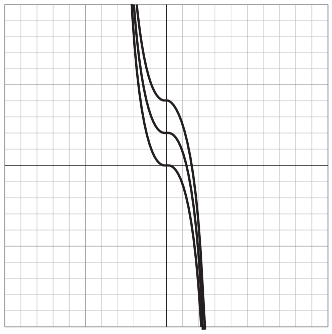
10 Find the absolute minimum and absolute maximum values of the function

$$f(x, y) = x^2 + 2y^2 - x$$

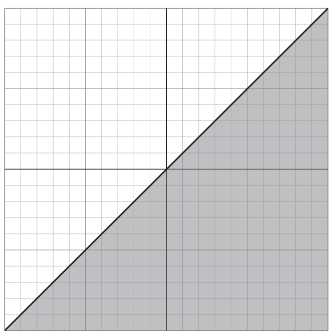
on the closed region $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ (the disk with radius one centered at the origin).

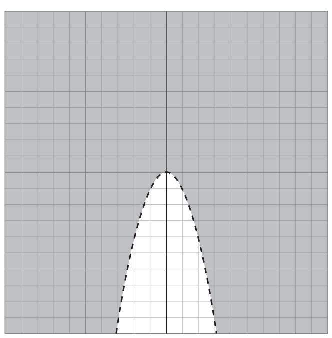
SOLUTIONS

1



2 The domain is the shaded gray region in each illustration.





3 (a) Restricted to the parametrized path $\langle t, 0\rangle$, this limit is

$$\lim_{t\to 0} \frac{t^2+0}{t^2+0} = 1$$

but along the parametrized path $\langle 0, t\rangle$, it is

$$\lim_{t\to 0} \frac{0+4t^2}{0+2t^2} = 2$$

Since the two values are different, the limit does not exist.

(b)

$$\lim_{(x,y)\to(0,0)} e^{x^2-2y^2} = e^0 = 1$$

4

$$\nabla f = \left\langle \frac{2}{2x+4y},\ \frac{4}{2x+4y},\ 2z\cos(z^2) \right\rangle.$$

5

$$f_x = \frac{1}{y^2} + y\cos(xy)$$

$$f_y = -\frac{2x}{y^3} + x\cos(xy)$$

$$f_{xx} = -y^2\sin(xy)$$

$$f_{yy} = \frac{6x}{y^4} - x^2\sin(xy)$$

7

$$\nabla f = \left\langle \frac{-2}{(2x+y)^2},\ \frac{-1}{(2x+y)^2} \right\rangle$$

$$\nabla f(1,1) = \left\langle -\frac{2}{9}, -\frac{1}{9} \right\rangle$$

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

$$f_{\mathbf{u}}(1,1) = \nabla f \cdot \mathbf{u} = -\frac{5}{9\sqrt{10}}$$

8

$$\nabla z = \langle 2x-2y, -2x+4y\rangle$$
$$\nabla z(1,2) = \langle -2, 6\rangle$$
$$\mathbf{u} = \frac{1}{\sqrt{40}}\langle -2, 6\rangle = \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

9

$$\nabla f = \langle 2x-y, 2y-x-7\rangle$$

The critical points are where $\nabla f = 0$, so

$$\begin{cases} 2x - y = 0 \\ 2y - x - 7 = 0 \end{cases}$$

From the first equation, we see that $y = 2x$. Plugging into the second, this means x=7/3, and so $y = 14/3$. The only critical point, then, is (7/3, 14/3).

10 First check for critical points in the interior.

$$\nabla f = \langle 2x - 1, 4y \rangle.$$

This is zero when $x = 1/2$ and $y = 0$, and the point $(1/2, 0)$ does lie in D. We can evaluate $f(1/2, 0) = -1/4$. Now we have to look along the boundary. The standard parametrization of this circle is $r(t) = \langle \cos t, \sin t \rangle$. Plugging in,

$$\begin{aligned} f(t) &= \cos^2 t + 2\sin^2 t - \cos t \\ f'(t) &= -2\sin t \cos t + 4 \sin t \cos t + \sin t \\ &= \sin t(2\cos t + 1) \end{aligned}$$

Setting $f'(t) = 0$, either $\sin t = 0$, or $\cos t = -1/2$, which occurs when $t = 0$, π, $2\pi/3$, and $4\pi/3$. These t values correspond to points $(1, 0)$, $(-1, 0)$, $(-1/2, \sqrt{3}/2)$, and $(-1/2, -\sqrt{3}/2)$ on the boundary circle. Plugging these points into f,

$$\begin{aligned} &f(1, 0) = 0 \\ &f(-1, 0) = 2 \\ &f(-1/2, \sqrt{3}/2) = 9/4 \\ &f(-1/2, -\sqrt{3}/2) = 9/4 \end{aligned}$$

Therefore the absolute minimum is $-1/4$ and the absolute maximum is $9/4$.