

# MATH 2050 TEST 1. SPRING 2013

**1** The first few terms of a sequence are

$$1, \quad -\frac{3}{2}, \quad \frac{5}{4}, \quad -\frac{7}{8}, \quad \frac{9}{16}, \quad -\frac{11}{32}, \quad \dots$$

Assuming this pattern continues, write a formula for  $a_n$ , the  $n^{\text{th}}$  term in this sequence, in terms of  $n$ .

**2** Compute the limits of the following sequences (if they exist).

$$(a) \quad \left\{ \frac{n+1}{\sqrt{4n^2-3}} \right\} \qquad (b) \quad \left\{ \frac{\ln(n)}{n^3} \right\}.$$

**3** Is the following sequence increasing, decreasing, or neither? [Give a complete argument—don't just look at the first few terms.]

$$\left\{ \frac{n+3}{2n+1} \right\}_{n=1}^{\infty}$$

**4** Evaluate the following (convergent) series

$$\sum_{n=1}^{\infty} \frac{4^{n-1}}{5^n}.$$

**5** Does the following series converge absolutely, converge conditionally, or does it diverge?

$$\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2-1}.$$

Problems 6-13 (8 points each) Do seven of the remaining eight problems. Indicate which one you do *not* want me to grade by placing an  $\times$  in the box. For each, determine whether the series converges or diverges. Show your work and explain your reasoning.

**6**   $\sum_{n=1}^{\infty} \frac{3n^2+4}{2n^3-n-1}$

**7**   $\sum_{n=1}^{\infty} \frac{e^n \cdot \pi^n}{n^n}$

**8**   $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$

**9**   $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{e^n}$

**10**   $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2+1}$

**11**   $\sum_{n=1}^{\infty} \left( \frac{\ln n}{n} \right)^n$

**12**   $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

**13**   $\sum_{n=0}^{\infty} \frac{2n}{n+1}$

SOLUTIONS

1  $a_n = (-1)^n \frac{2n+1}{2^n}$

2 (a)

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n+1}{\sqrt{4n^2-3}} \cdot \frac{1/n}{1/\sqrt{n^2}} \\ &= \lim_{n \rightarrow \infty} \frac{1+(1/n)}{\sqrt{4-(3/n^2)}} \\ &= 1/2. \end{aligned}$$

(b)

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\ln n}{n^3} &= \lim_{x \rightarrow \infty} \frac{\ln x}{x^3} \\ &\stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{3x^2} \\ &= \lim_{x \rightarrow \infty} \frac{1}{3x^3} \\ &= 0. \end{aligned}$$

3 Look at the continuous function  $f(x) = (x+3)/(2x+1)$ . Then

$$\begin{aligned} f'(x) &= \frac{(2x+1) \cdot 1 - (x+3) \cdot 2}{(2x+1)^2} \\ &= \frac{-5}{(2x+1)^2} \\ &< 0. \end{aligned}$$

Therefore  $f(x)$  is decreasing, so the sequence is decreasing as well.

4

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{4^{n-1}}{5^n} &= \sum_{n=0}^{\infty} \frac{4^n}{5^{n+1}} \\ &= \frac{1}{5} \sum_{n=0}^{\infty} (4/5)^n \\ &= \frac{1}{5} \cdot \frac{1}{1-(4/5)} \\ &= 1. \end{aligned}$$

5 First use the alternating series test. The limit of the terms is zero:

$$\lim_{n \rightarrow \infty} \frac{n}{n^2-1} = \lim_{n \rightarrow \infty} \frac{1}{n-(1/n)} = 0;$$

and the sequence is decreasing because if we put  $f(x) = x/(x^2-1)$ , then

$$f'(x) = \frac{(x^2-1) - x(2x)}{(x^2-1)^2} = \frac{-1-x^2}{(x^2-1)^2} < 0.$$

Therefore the series converges.

Now note that

$$\frac{n}{n^2-1} > \frac{1}{n}$$

and  $\sum(1/n)$  diverges ( $p$ -test,  $p=1$ ). By the basic comparison test, then, the series  $\sum n/(n^2-1)$  diverges. Therefore the series converges conditionally.

6 Use the limit comparison test, comparing to  $\sum(1/n)$ . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{3n^2+4}{2n^3-n+1} \cdot \frac{n}{1} \\ &= \lim_{n \rightarrow \infty} \frac{3n^3+4n}{2n^3-n-1} \\ &= \lim_{n \rightarrow \infty} \frac{3+(4/n^2)}{2-(1/n^2)-(1/n^3)} \\ &= 3/2. \end{aligned}$$

The series  $\sum(1/n)$  diverges ( $p$ -test,  $p=1$ ), so this series diverges as well.

7 Use the root test

$$\lim_{n \rightarrow \infty} \left| \frac{e^n \pi^n}{n^n} \right|^{1/n} = \lim_{n \rightarrow \infty} \frac{e\pi}{n} = 0 < 1.$$

The series converges.

8 Use the ratio test

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(n+1)!(n+2)!}{(2n+2)!} \cdot \frac{(2n)!}{n!(n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)(n+2)}{(2n+2)(2n+1)} \\ &= 1/4 < 1. \end{aligned}$$

The series converges.

9 Use the root test

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n n}{e^n} \right|^{1/n} = \lim_{n \rightarrow \infty} \frac{n^{1/n}}{e} = 1/e < 1.$$

The series converges.

**10** Use the basic comparison test, comparing this to  $\sum 1/n^2$ . Note that

$$\frac{\sin^2(n)}{n^2 + 1} < \frac{1}{n^2}$$

and  $\sum 1/n^2$  converges ( $p$ -test,  $p = 2 > 1$ ). Therefore this series converges.

**11** Use the root test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \left( \frac{\ln n}{n} \right)^n \right|^{1/n} &= \lim_{n \rightarrow \infty} \frac{\ln n}{n} \\ &= \lim_{x \rightarrow \infty} \frac{\ln x}{x} \\ &= \lim_{x \rightarrow \infty} \frac{1/x}{1} \\ &= 0 < 1. \end{aligned}$$

The series converges.

**12** Use the integral test. By the substitution  $u = \ln x$ ,  $du = (1/x)dx$ ,

$$\int \frac{dx}{x(\ln x)^3} = \int u^{-3} du = -\frac{1}{2u^{-2}} = -\frac{1}{2(\ln x)^2}$$

and so

$$\begin{aligned} \int_2^\infty \frac{dx}{x(\ln x)^3} &= \lim_{a \rightarrow \infty} \int_2^a \frac{dx}{x(\ln x)^3} \\ &= \lim_{a \rightarrow \infty} \left( -\frac{1}{2(\ln x)^2} \right) \Big|_2^a \\ &= \lim_{a \rightarrow \infty} \left( -\frac{1}{2(\ln a)^2} + \frac{1}{2(\ln 2)^2} \right) \\ &= \frac{1}{2(\ln 2)^2} \end{aligned}$$

**13** Use the divergence test:

$$\lim_{n \rightarrow \infty} \frac{2n}{n+1} = \lim_{n \rightarrow \infty} \frac{2}{1 + (1/n)} = 2 \neq 0.$$

The series diverges.