Math 3060 Test 2. Fall 2013

1 $\sum_{k=1}^{\infty} \frac{x^k}{k^3}$

2 $\sum_{k=1}^{\infty} \frac{(x-1)^k}{k \cdot 2^k}$

3 Recall the definition of the Taylor series of a function,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Derive the Taylor series for the function $f(x) = \sin x$, expanded around the point $c = \pi/2$.

Questions 4 – 9. Find the Taylor series for the given function (expanded about $c = 0$) using a known series. Your answer should be a single summation of the form $\sum a_n x^n$. Be sure to give the interval of convergence.

4 $f(x) = \dfrac{1}{1-3x}$

5 $f(x) = x\cos(2x^3)$

6 $f(x) = \ln(1-4x)$

7 $f(x) = e^{x+3}$

8 $f(x) = \dfrac{1}{(2+x)^2}$

9 $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$

10 Use Taylor series to evaluate the following limit

$$\lim_{x \to 0} \frac{2e^x - 2 - 2x - x^2}{\cos x - 1}.$$

11 Give a series solution to the following integral

$$\int \frac{\cos(2x) - 1}{x^2}\, dx$$

12 Use a Taylor series to approximate $\cos(1/10)$ to within a margin of error of $1/10,000$. Be sure to explain how you know your approximation is sufficiently accurate.

13 What the sum of the following series (that is, what does it actually add up to)?

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{16^n (2n)!}$$

14 Use the Taylor series for $f(x) = e^x$ to demonstrate that $\dfrac{d}{dx}(e^x) = e^x$.

1 First use the root test:

$$\lim_{k\to\infty}\left|\frac{x^k}{k^3}\right|^{1/k} = |x| < 1 \implies -1 < x < 1.$$

Now test the endpoints. When $x = 1$, the series $\sum 1/k^3$ converges absolutely. Therefore the power series converges at both $x = 1$ and $x = -1$, and so the interval of convergent is $[-1, 1]$.

2 Use the root test:

$$\lim_{k\to\infty}\left|\frac{(x-1)^k}{k\cdot 2^k}\right|^{1/k} = \frac{|x-1|}{2} < 1$$

For the inequality to be true, we must have $-2 < |x-1| < 2$; that is, $-1 < x < 3$. Now test the endpoints. At $x = -1$, the series is

$$\sum\frac{(-2)^k}{k\cdot 2^k} = \sum\frac{(-1)^k}{k}$$

which converges by the Alternating Series Test. At $x = 3$, the series is

$$\sum\frac{2^k}{k\cdot 2^k} = \sum\frac{1}{k}$$

which diverges (it is the harmonic series). Therefore the interval of convergence is $[-1, 3)$.

3 First construct a table of derivatives of $f(x)$ at $\pi/2$.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
5	$\cos x$	0
6	$-\sin x$	-1
7	$-\cos x$	0

The Taylor series is then

$$\frac{1}{0!}\left(x-\frac{\pi}{2}\right)^0 - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 - \cdots$$

or in summation notation:

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}$$

4

$$f(x) = \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty}3^n x^n$$

The interval of convergence is $(-1/3, 1/3)$.

5

$$\begin{aligned} f(x) &= x\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2x^3)^{2n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n 4^n}{(2n)!}x^{6n+1} \end{aligned}$$

The interval of convergence is $(-\infty, \infty)$.

6

$$\begin{aligned} \ln(1-4x) &= \sum_{n=0}^{\infty}(-1)^n\frac{(-4x)^{n+1}}{n+1} \\ &= \sum_{n=0}^{\infty}\frac{(-1)4^{n+1}}{n+1}x^{n+1} \end{aligned}$$

The interval of convergence is $(-1/4, 1/4)$.

7

$$f(x) = e^3 e^x = e^3\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{e^3 x^n}{n!}$$

The interval of convergence is $(-\infty, \infty)$.

8

$$\begin{aligned} \int f(x)\,dx &= -\frac{1}{2+x} \\ &= -\frac{1}{2}\cdot\frac{1}{1+(x/2)} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty}(-x/2)^n \\ &= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1}x^n \end{aligned}$$

The Taylor series for $f(x)$ is the derivative of this:

$$f(x) = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n+1} nx^{n-1}$$

The interval of convergence is $(-2, 2)$.

9

$$\begin{aligned} f(x) &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!}\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}[1 + (-1)^{n+1}]\frac{x^n}{n!} \\ &= \frac{1}{2}\sum_{k=0}^{\infty} 2\frac{x^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

The interval of convergence is $(-\infty, \infty)$.

10

$$\begin{aligned} &\lim_{x\to 0} \frac{2e^x - 2 - 2x - x^2}{\cos x - 1} \\ &= \lim_{x\to 0} \frac{2\left[\sum_{n=0}^{\infty} x^n/n!\right] - 2 - 2x - x^2}{\left[\sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!\right] - 1} \\ &= \lim_{x\to 0} \frac{2[1 + x + x^2/2 + \cdots] - 2 - 2x - x^2}{[1 - x^2/2 + x^4/24 - \cdots] - 1} \\ &= \lim_{x\to 0} \frac{2x^3/6 + \cdots}{-x^2/2 + \cdots} \\ &= 0 \end{aligned}$$

11

$$\begin{aligned} &\int \frac{\cos(2x) - 1}{x^2}\,dx \\ &= \int \frac{\left[\sum_{n=0}^{\infty}(-1)^n/(2n)! \cdot (2x)^{2n}\right] - 1}{x^2}\,dx \\ &= \int \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n} \cdot x^{2n-2}\,dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} \cdot \frac{x^{2n-1}}{2n-1} + C \end{aligned}$$

12

$$\begin{aligned} \cos(1/10) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{10}\right)^{2n} \\ &= 1 - \frac{1}{2!}\cdot\frac{1}{10^2} + \frac{1}{4!}\cdot\frac{1}{10^4} - \cdots \\ &\approx 1 - \frac{1}{200} = \frac{199}{200} \end{aligned}$$

Note that the series is alternating, hence the nth partial sum differs from the actual infinite sum by less than $|a_{n+1}|$. In this case, the *third* term in the series is less than 1/10,000, so the sum of the first *two* terms is sufficiently accurate.

13

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{16^n (2n)!} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{4}\right)^{2n} \\ &= \cos(\pi/4) \\ &= \sqrt{2}/2 \end{aligned}$$

14

$$\begin{aligned} \frac{d}{dx}e^x &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{1}{n!}x^n\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!}nx^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}x^n \\ &= e^x \end{aligned}$$