Math 3060 Test 3. Fall 2013

1 What is the distance between the points $(2, 3, 8)$ and $(-1, 0, 6)$?

2 What is the length of the vector $\mathbf{v} = \langle 2, 1, 3\rangle$? What is the unit vector in the same direction as $\mathbf{v}$?

3 If $\mathbf{v} = \langle 2, 1, 4\rangle$ and $\mathbf{w} = \langle -3, -2, 5\rangle$ what is $\mathbf{v} \cdot \mathbf{w}$?

4 If $\mathbf{v} = \langle 1, 1, -2\rangle$ and $\mathbf{w} = \langle 2, 1, -3\rangle$, what is $\mathbf{v} \times \mathbf{w}$?

5 Give parametric equations for the line which passes through the points $(2, 1, 5)$ and $(4, 3, 2)$.

6 What is the equation of the plane which passes through the points $(2, 1, 5)$, $(4, 3, 2)$, and $(1, 0, 3)$?

7 Give parametric equations for the line of intersection of the two planes:

$$x + y + 3z = 0 \quad \text{and} \quad x - y + 2z = 1.$$

8 Sketch the graphs of each of the following surfaces:

(a) $z^2 = x^2 + y^2$

(b) $z = x^2$

9 Evaluate the limit:

$$\lim_{t\to 1} \left\langle \frac{\sin(\pi t)}{t},\ \ln(t),\ e^{(t^2-1)/(t-1)} \right\rangle.$$

10 Compute the derivative of the following vector-valued function

$$r(t) = \left\langle \sin^2(t),\ \frac{1}{\ln(t)},\ \frac{1}{t^2} \right\rangle.$$

11 For the parametric curve $r(t) = \langle \cos(2t), \sin(2t), t\rangle$, compute $T(\pi/4)$, $N(\pi/4)$ and $B(\pi/4)$, the unit tangent, unit normal, and binormal vectors.

12 Compute the arc length of the parametric curve

$$r(t) = \left\langle e^t \cos t, 2, e^t \sin t \right\rangle, \quad 0 \le t \le \pi.$$

SOLUTIONS

1 $d = \sqrt{3^2 + 3^2 + 2^2} = \sqrt{22}$

2 $|\mathbf{v}| = \sqrt{4+1+9} = \sqrt{14}$, so the unit vector in the same direction as $\mathbf{v}$ is

$$\mathbf{u} = \mathbf{v}/\sqrt{14} = \left\langle \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle.$$

3 $\mathbf{v} \cdot \mathbf{w} = 2(-3) + 1(-2) + 4(5) = 12$

4

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ 2 & 1 & -3 \end{vmatrix} = \langle -1, -1, -1 \rangle.$$

5 $r_0 = \langle 2, 1, 5 \rangle$ and $d = \langle 2, 2, -3 \rangle$. The line is given by the equation

$$r(t) = r_0 + td = \langle 2 + 2t, 1 + 2t, 5 - 3t \rangle.$$

6 Form vectors between pairs of points, for instance $v_1 = \langle 2, 2, 3 \rangle$ and $v_2 = \langle -1, -1, -2 \rangle$. The normal to the plane is

$$n = v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 2 & 2 & 3 \\ -1 & -1 & -2 \end{vmatrix} = \langle -1, 1, 0 \rangle.$$

The equation of the plane is then

$$\langle -1, 1, 0 \rangle \cdot \langle x - 2, y - 1, z - 5 \rangle = 0$$
$$-(x - 2) + (y - 1) = 0$$
$$y - x = -1$$

7 Adding the two equations together gives

$$2x + 5z = 1.$$

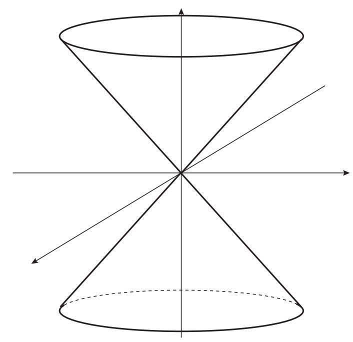
That is, $z = (1 - 2x)/5$. Now plug that into the first equation to find

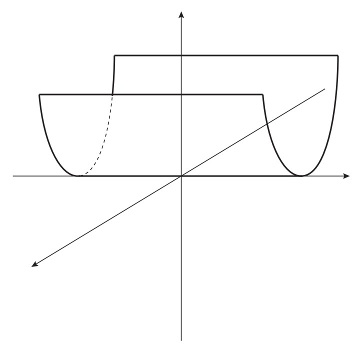
$$x + y + 3 \cdot \frac{1 - 2x}{5} = 0 \implies y = \frac{x - 3}{5}$$

The intersection line can then by parametrized by

$$r(t) = \left\langle t, \frac{t-3}{5}, \frac{1-2t}{5} \right\rangle.$$

8 The first surface is a cone. The second is a parabolic cylinder.





9

$$\lim_{t\to 1} \left\langle \frac{\sin(\pi t)}{t},\ \ln(t),\ e^{(t^2-1)/(t-1)} \right\rangle$$
$$= \left\langle \lim_{t\to 1} \frac{\sin(\pi t)}{t},\ \lim_{t\to 1} \ln(t),\ \lim_{t\to 1} e^{(t^2-1)/(t-1)} \right\rangle$$
$$= \langle 0, 0, e^2 \rangle$$

10

$$r'(t) = \left\langle 2 \sin t \cos t,\ -(\ln t)^{-2} \cdot \frac{1}{t},\ -\frac{2}{t^3} \right\rangle.$$

11 Start with the derivative

$$\begin{aligned} r'(t) &= \langle -2\sin(2t), 2\cos(2t), 1\rangle \\ |r'(t)| &= \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 1} = \sqrt{5} \\ T(t) &= \left\langle -\frac{2}{\sqrt{5}}\sin(2t), \frac{2}{\sqrt{5}}\cos(2t), \frac{1}{\sqrt{5}} \right\rangle \\ &\implies T(\pi/4) = \left\langle -2/\sqrt{5}, 0, 1/\sqrt{5} \right\rangle \\ T'(t) &= \left\langle -\frac{4}{\sqrt{5}}\cos(2t), -\frac{4}{\sqrt{5}}\sin(2t), 0 \right\rangle \end{aligned}$$

To simplify, note that we can factor out a $4/\sqrt{5}$ to get a parallel vector

$$\langle -\cos(2t), -\sin(2t), 0\rangle$$

This is a unit vector, so this is $N(t)$. Therefore

$$N(\pi/4) = \langle 0, -1, 0\rangle.$$

Finally,

$$\begin{aligned} B(\pi/4) &= \begin{vmatrix} i & j & k \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & -1 & 0 \end{vmatrix} \\ &= \left\langle \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \right\rangle. \end{aligned}$$

12

$$\begin{aligned} r'(t) &= \langle -e^t\sin t + e^t\cos t, 0, e^t\cos t + e^t\sin t\rangle \\ |r'(t)| &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2} \\ &= e^t\sqrt{\cos^2 t + \sin^2 t + \cos^2 t + \sin^2 t} \\ &= \sqrt{2}e^t \end{aligned}$$

The arc length is then

$$s = \int_0^\pi |r'(t)|\, dt = \int_0^\pi \sqrt{2}e^t\, dt = \sqrt{2}e^{\pi]} - \sqrt{2}.$$