Math 3060 Test 4. Fall 2013

1. Sketch the domains of the following functions.

1. $f(x,y) = \sqrt{1-2x-3y}$

2. $f(x,y) = \sqrt{x} + \sqrt{y}$

2. Evaluate the following limits or demonstrate that they do not exist.

1. $\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^2-y^2}{x-y}\right)$

2. $\lim_{(x,y)\to(0,0)} \frac{x^2+y}{x+y}$

3. For the function $f(x,y) = x^2e^y + \sin(xy)$

1. Compute f_x.
2. Compute f_y.

4. Suppose that f is a function of x, y, and z, and that x, y, and z are each functions of s and t. Write out a "chain rule" formula to compute $\partial f/\partial s$.

5. Compute the gradient of the function

$$F(x,y) = x^2\cos y + y^3$$

at the point $(1,\pi)$.

6. Give a unit vector (in the xy-plane) which points in the direction of greatest increase of the function $f(x,y) = x^3 - 2xy^3$ at the point $(1,1)$.

7. Find the equation of the tangent plane to the surface $z = x^3 - xy^2$ at the point $(1,1,0)$. Note that we have discussed two ways to do this– you may use either method.

8. Find all the critical points of the function $f(x,y) = x^2 + xy + y^2 + y$.

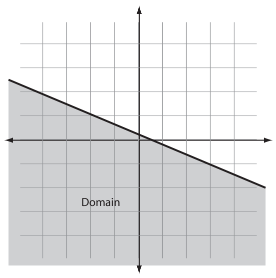
9. Find the absolute maximum and minimum values of the function $f(x,y) = 2x^3 + y^4$ on the closed domain

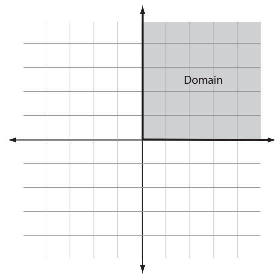
$$D = \left\{(x,y) \;\middle|\; x^2+y^2 \le 1\right\}.$$

10. What are the dimensions of the box with largest volume that has three of its faces on the coordinate planes, and one vertex on the plane $x + 2y + 4z = 4$?

SOLUTIONS

1 For the first, we need $2x + 3y \leq 1$. For the second, we need both $x \geq 0$ and $y \geq 0$.





2 1. Since cosine is a continuous function, we may first evaluate the limit inside the function, and then plug in the result:

$$\begin{aligned}
&\cos\left(\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x - y}\right) \\
&= \cos\left(\lim_{(x,y)\to(0,0)} (x + y)\right) \\
&= \cos 0 \\
&= 1.
\end{aligned}$$

2. We can see that this limit does not exist by comparing the value it approaches along two different paths: along the line $\langle t, 0\rangle$, we get

$$\lim_{t\to 0} \frac{t^2}{t} = \lim_{t\to 0} t = 0;$$

but along the line $\langle 0, t\rangle$, we get

$$\lim_{t\to 0} \frac{t}{t} = 1.$$

The values are not equal, so the limit does not exist.

3

$$\begin{aligned}
f_x &= 2xe^y + y\cos(xy) \\
f_y &= x^2 e^y + x\cos(xy)
\end{aligned}$$

4

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

5

$$\begin{aligned}
\nabla F &= \langle 2x\cos y, -x^2 \sin y + 3y^2\rangle \\
\nabla F(1,\pi) &= \langle -2, 3\pi^2\rangle
\end{aligned}$$

6 The gradient points in the direction of greatest increase. So

$$\nabla f = \langle 3x^2 - 2y^3, -6xy^2\rangle$$

At the point $(1, 1)$, we have $\nabla f(1, 1) = \langle 1, -6\rangle$. The *unit* vector in this direction is

$$\frac{\langle 1, -6\rangle}{\sqrt{1^2 + 6^2}} = \left\langle \frac{1}{\sqrt{37}}, \frac{-6}{\sqrt{37}} \right\rangle.$$

7 Think of this surface as a level surface of the function $F(x, y, z) = z + xy^2 - x^3$. The gradient ∇F is perpendicular to this level surface and will serve as the normal vector to the tangent plane.

$$\begin{aligned}
\nabla F &= \langle y^2 - 3x^2, 2xy, 1\rangle \\
n = \nabla F(1, 1, 0) &= \langle -2, 2, 1\rangle
\end{aligned}$$

The equation of the tangent plane is then

$$\begin{aligned}
\langle -2, 2, 1\rangle \cdot \langle x - 1, y - 1, z\rangle &= 0 \\
-2(x - 1) + 2(y - 1) + z &= 0 \\
-2x + 2y + z &= 0
\end{aligned}$$

8 Compute the gradient:

$$\nabla F = \langle 2x + y, x + 2y + 1\rangle.$$

It is defined everywhere, so we need to find out when it is zero:

$$\begin{cases} 2x + y = 0 \\ x + 2y + 1 = 0 \end{cases}$$

The only solution to this system of equations is the point $(1/3, -2/3)$.

9 First compute the gradient to find critical points in the interior of D:

$$\nabla f = \langle 6x^2, 4y^3\rangle,$$

so $\nabla f = 0$ at the point $(0,0)$. Now parametrize the edge as

$$r(t) = \langle \cos t, \sin t \rangle.$$

Then

$$f(t) = 2\cos^3 t + \sin^4 t$$

which has the derivative

$$\begin{aligned} f'(t) &= -6\cos^2 t \sin t + 4\sin^3 t \cos t \\ &= 2\sin t \cos t(2\sin^2 t - 3\cos t) \end{aligned}$$

This is defined for all t, so the critical points will be where it is zero.

$$\begin{aligned} &2\sin t\cos t(2(1-\cos^2 t) - 3\cos t) = 0 \\ &-2\sin t \cos t(2\cos^2 t + 3\cos t - 2) = 0 \\ &-2\sin t\cos t(2\cos t - 1)(\cos t + 2) = 0 \end{aligned}$$

Setting each piece equal to zero, we get a lot of critical points: at $t = 0$, π, $\pi/2$, $3\pi/2$, $\pi/3$, and $5\pi/3$. Those, together with the point $(0,0)$ are the critical points of f. Now plug them in to find the absolute max and min.

$$\begin{aligned} &f(0,0) = 0 \\ &f(1,0) = 2 \\ &f(-1,0) = -2 \\ &f(0,\pm 1) = 1 \\ &f(1/2, \pm\sqrt{3}/2) = 13/16 \end{aligned}$$

The absolute max is 2, and the absolute min is -2.

10 We need to maximize $V = xyz$ subject to the constraint $x = 4 - 2y - 4z$.

$$\begin{aligned} V &= (4 - 2y - 2z)yz \\ &= 4yz - 2y^2z - 4yz^2 \\ \nabla V &= \langle 4z - 4yz - 4z^2, 4y - 2y^2 - 8yz \rangle \\ &= \langle 4z(1 - y - z), 2y(2 - y - 4z) \rangle. \end{aligned}$$

This can be zero if one or both of y and z are zero, but those values will not give a box of *maximum* volume. The maximum volume will happen when

$$\begin{cases} y + z = 1 \\ y + 4z = 2 \end{cases}$$

The only solution to this system is $z = 1/3$ and $y = 2/3$. In this case, the corresponding x value is $4/3$. The box with maximum volume has dimensions

$$\frac{4}{3} \times \frac{2}{3} \times \frac{1}{3}.$$