Math 3350. Final Exam (Harvey Spring 2012).

Name (4 points): ____

[No books or notes are allowed. Please show all of your work.]

1. (4 points) Define: Let G and H be groups. A map $f : G \to H$ is an *isomorphism* if:

2. (4 points) Define: A permutation σ is *even* if:

3. (4 points) Define: A subgroup H of G is a *normal* subgroup if:

4. (4 points) Define: Let a be an element in a group G. The *order* of a is:

5. (4 points) Define: A group G is *cyclic* if:

6. (4 points) Define: the *kernel* of a homomorphism $f : G \to H$ is:

7. (4 points) An element $g \in G$ is in $Z(G)$, the center of G if:

8. (4 points) State Lagrange's Theorem.

9. (6 points) Give the order of each of the following groups.

(a) $\mathbb{Z}_{15}$

(b) $U(15)$

(c) S_5

(d) A_5

(e) $\mathbb{Z}_2 \oplus S_3$

(f) $\mathbb{Z}_{18}/\langle 3\rangle$

10. (6 points) Give the order of each of the following elements.

(a) $9 \in \mathbb{Z}_{12}$

(b) $3 \in U(8)$

(c) $(123) \in S_3$

(d) $(23)(4567) \in S_7$

(e) $(2,3) \in \mathbb{Z}_4 \oplus \mathbb{Z}_9$

(f) $8 + \langle 6\rangle \in \mathbb{Z}_{12}/\langle 6\rangle$

11. (4 points) Let $\sigma = (12356)(234)$.

(a) Write σ as a product of disjoint cycles.

(b) Write σ as a product of 2-cycles (transpositions)

12. (4 points) List all abelian groups of order 98 up to isomorphism (do not include any "repeats").

13. (4 points) (a) Give an example of a homomorphism which is one-to-one but not onto.

(b) Give an example of a homomorphism which is onto but not one-to-one.

14. (4 points) List the left cosets of the subgroup $\langle 4 \rangle$ in $\mathbb{Z}_{20}$.

15. (4 points) (a) Give an example of a finite non-abelian group.

(b) Give an example of an infinite abelian group that is not cyclic.

16. (4 points) According to the Fundamental Theorem of Finite Abelian Groups, $U(12)$ is isomorphic to a direct product of cyclic groups of prime power order. Which one?

17. (4 points) Draw the subgroup lattice of $\mathbb{Z}_{18}$.

18. (8 points) Let g be a fixed element in a group G. Define the set

$$H = \{h \in G | hg = gh\}.$$

Prove that H is a subgroup of G (it is called the centralizer of g).

19. (8 points) Define a map $f : S_6 \to S_6$ by

$$f(x) = (12)\,x\,(12).$$

Prove that f is an automorphism of S_6.

20. (8 points) Let p be a prime and suppose that G is an abelian group with order p^2. Prove that if G has at least p elements with order p, then $G \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$.