Math 3350. Final Exam (Harvey Spring 2013).

Name (2 points): ______________________________

[No books, notes, or calculators are allowed. Please show all of your work.]

Definitions. Be precise and concise. Do not give vague answers.

1. (4 points) A set S with a closed binary operation $\circ$ is a *group* if:

2. (4 points) Let H be a nonempty subset of group G. What are the two steps of the two-step subgroup test to show that H is a subgroup of G?

3. (4 points) A subgroup H of a group G is *normal* if:

4. (4 points) The center $Z(G)$ of a group G is:

5. (4 points) State the Fundamental Theorem of Finite Abelian Groups.

6. (4 points) State Lagrange's theorem.

7. (4 points) Suppose that G and H are groups. A map $f : G \to H$ is a *group homomorphism* if:

Problems. Short answers are sufficient.

8. (4 points) (a) Give an example of a finite abelian group.

   (b) Give an example of a finite non-cyclic group.

9. (4 points) (a) Give an example of a finite subgroup of an infinite group (other than the identity).

   (b) Give an example of a infinite subgroup of an infinite group (other than the group itself).

10. (4 points) Give an example of a normal subgroup of a non-abelian group.

11. (4 points) Let $\sigma = (1234)(568)$ and $\tau = (145)(2367)$. Write $\sigma \cdot \tau$ as a product of disjoint cycles.

12. (8 points) What are the orders of the following groups?

(a) S_4

(a) $\mathbb{Z}_6 \oplus \mathbb{Z}_3$

(b) S_5/A_5

(c) $A_4 \oplus U(4)$

13. (8 points) What are the orders of the following elements?

(a) 2 in Z_9

(b) (123)(456) in S_6

(c) $((123), 2)$ in $S_3 \oplus \mathbb{Z}_{12}$

(d) eH in G/H (where $H \lhd G$ and e is the identity in G).

14. (4 points) Draw the subgroup lattice of $\mathbb{Z}_{36}$.

15. (4 points) List, up to isomorphism, all abelian groups of order 36. List each exactly once– no repeats!

Proofs. Give clear arguments. Don't skip steps.

16. (8 points) Let H be a subgroup of G and let N be a subgroup of H that is normal in G. Prove that the set

$$H/N = \left\{ hN \in G/N \,\middle|\, h \in H \right\}$$

is a subgroup of G/N.

17. (8 points) Let G be the set of integers together with the operation $a \circ b = a + b - 1$ (we proved on Friday that this is a group). Prove that the map $f : G \to \mathbb{Z}$ defined by $f(g) = g - 1$ is a group isomorphism.

18. (8 points) Let Δ be the subset of $G \oplus G$ which consists of elements with the same coordinates. That is,

$$\Delta = \left\{ (g_1, g_2) \in G \oplus G \,\middle|\, g_1 = g_2 \right\}.$$

Prove that Δ is a normal subgroup of $G \oplus G$, and that $(G \oplus G)/\Delta \simeq G$. [Hint: the quick way to do this is to construct the right homomorphism from $G \oplus G$ onto G and to use the First Isomorphism Theorem.]

19. (8 points) Suppose that G is an abelian group with order 40 that has at least one element of order 20 and at least two elements of order 2. Prove that $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5$.

I certify as a student at The University of Virginia's College at Wise that I have neither received nor given aid on this test.

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