REVIEW

1. Give the equation of the plane containing the line $r(t) = \langle 2-t, 1+2t, 3+t\rangle$ and the point $(2,0,1)$. Also, parametrize that plane.

2. Give parametric equations for the line which intersects $r(t) = \langle 1+t, 2-t, 3+2t\rangle$ at the point $(1,2,3)$ and is perpendicular to that line.

3. Sketch the regions:

   (cylindrical) $0 \le r \le 2$, $0 \le \theta \le \pi/2$, $0 \le z \le 2-r$

   (spherical) $0 \le \rho \le 1$, $\pi/4 \le \theta \le \pi/2$, $0 \le \phi \le \pi/2$.

4. Convert the cylindrical equation $z = \sqrt{4-r^2}$ to rectangular and spherical coordinates.

5. For what value or values of k does the following limit exist:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2+5y^2}{x^2+ky^2}$$

6. Show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

7. (problem 33 in §2.3) Given $f(s,t) = (s^2, st, t^2)$, and $\mathbf{a} = (-1,1)$, compute $Df(\mathbf{a})$.

8. (problem 29b in §2.4) Show that $T(x,y,t) = e^{-kt}(\cos x + \cos y)$ satisfies the two-dimensional heat equation

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \frac{\partial T}{\partial t}.$$

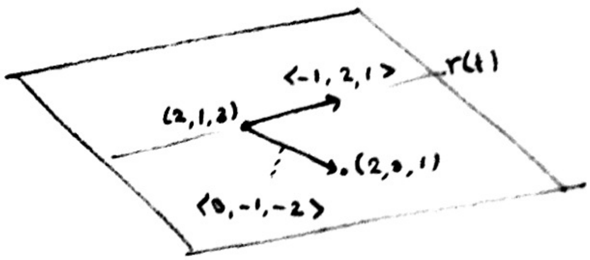
9. (problem 21 in §2.5) Let $f(x,y) = ye^x$ and $g(s,t) = (s-t, s+t)$. Compute $D(f \circ g)$.

10. Let $f(x,y,z) = x^2 - xz + z^2$, and $x = 2s+t$, $y = s^2 - t$, $z = t^2$. Use the chain rule to $\partial f/\partial s$ and $\partial f/\partial t$ in two ways: first using the tree diagram method, and second by multiplying matrices.

11. Find the equation for the tangent plane to the surface $z = x^2 - 3y^2$ at the point $(1,1,-2)$ in two ways: first using the partial derivative to form vectors in the tangent plane, and second by viewing this as a level surface of a function of three variables and using the gradient.

12. Compute the directional derivative of $f(x,y) = x^2 + 3xy - y^2$ at the point $(2,3)$ in the direction parallel to $\langle 1,2\rangle$. What is the unit vector which points in the direction of greatest increase at this point?

1.



$$\underline{n} = \begin{vmatrix} i & j & k \\ -1 & 2 & 1 \\ 0 & -1 & -2 \end{vmatrix} = \langle -3, -2, 1 \rangle$$

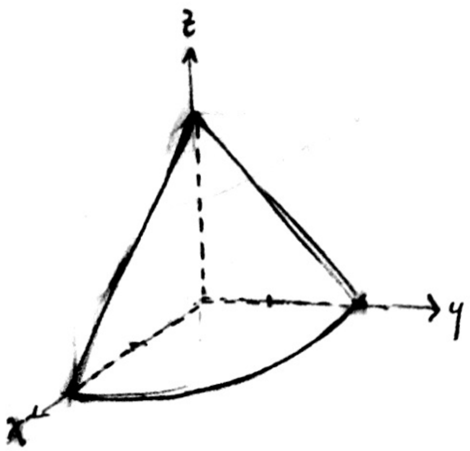
Eqⁿ of plane: $\langle -3, -2, 1 \rangle \cdot \langle x-2, y, z-1 \rangle = 0$

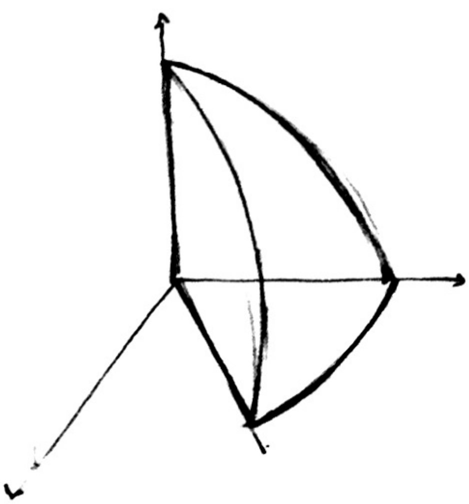
$$-3x - 2y + z = -5.$$

2. need a direction vector perp. to $\langle 1, -1, 2 \rangle$: $\underline{d} = \langle 1, 1, 0 \rangle$ works (dot product is zero). Then

$$r(t) = \langle 1, 2, 3 \rangle + t \langle 1, 1, 0 \rangle$$
$$= \langle 1+t, 2+t, 3 \rangle.$$

3.





4. $z = \sqrt{4 - x^2 - y^2}$: $z^2 = 4 - x^2 - y^2$

$$x^2 + y^2 + z^2 = 4.$$

$$\rho^2 = 4$$

$\rho = 2$ (its only top half of sphere so restrict to $0 \le \phi \le \pi/2$).

5. If $k = 5/2$, then

$$\lim_{(x,y)\to(0,0)} \frac{2x^2+5y^2}{x^2+\frac{5}{2}y^2} = \lim_{(x,y)\to(0,0)} \frac{2(x^2+5/2\, y^2)}{x^2+5/2\, y^2} = 2.$$

If $k \neq 5/2$, then:

along path $\langle t, 0\rangle$: $\lim_{t\to 0} \frac{2t^2}{t^2} = 2.$

along path $\langle 0, t\rangle$: $\lim_{t\to 0} \frac{5t^2}{kt^2} = \frac{5}{k} \neq 2.$

so limit does not exist.

6. Along the path $\langle t, 0\rangle$,

$$\lim_{t\to 0} \frac{t\cdot 0^2}{t^2+0^4} = 0.$$

limit D.N.E.

Along the path $\langle t^2, t\rangle$,

$$\lim_{t\to 0} \frac{t^2\cdot t^2}{(t^2)^2+t^4} = \lim_{t\to 0} \frac{t^4}{2t^4} = \frac{1}{2}$$

7. $$Df = \begin{bmatrix} 2s & 0 \\ t & s \\ 0 & 2t \end{bmatrix} \Rightarrow Df(-1,1) = \begin{bmatrix} -2 & 0 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

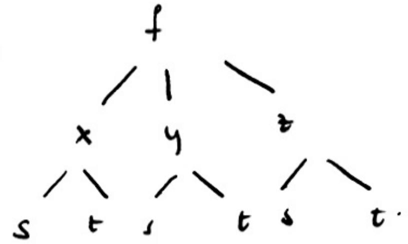
8. $\frac{\partial T}{\partial x} = -e^{-kt}\cdot \sin x$; $\frac{\partial^2 T}{\partial x^2} = -e^{-kt}\cos x$

$\frac{\partial T}{\partial y} = -e^{-kt}\cdot \sin y$; $\frac{\partial^2 T}{\partial y^2} = -e^{-kt}\cos y$

$$\frac{\partial T}{\partial t} = -ke^{-kt}(\cos x+\cos y) = k\left(-e^{-kt}\cos x - e^{-kt}\cos y\right) = k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right).$$

9.
$$D(f\circ g) = Df \cdot Dg$$
$$= \left[ye^x,\ e^x\right]\cdot\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \left[ye^x + e^x,\ -ye^x + e^x\right]$$

10.



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$
$$= (2x-z)\cdot 2 + (0) + (-x+2z)(0)$$
$$= 4x-2z$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t}$$
$$= (2x-z)\cdot(1) + (0) + (-x+2z)(2t)$$
$$= 2x - z - 2xt + 4zt$$

$\frac{\partial f}{\partial s}$

$$Df = \begin{bmatrix}2x-z & 0 & -x+2z\end{bmatrix}\cdot\begin{bmatrix}2 & 1\\ 2s & -1\\ 0 & 2t\end{bmatrix} = \begin{bmatrix}2(2x-z)+0+0\\ (2x-z)-(0)+(-x+2z)2t\end{bmatrix}$$

$\frac{\partial f}{\partial t}$

11. (a)

$$\frac{\partial z}{\partial x} = 2x \qquad \underline{\text{at } (1,1,-2)} \qquad \frac{\partial z}{\partial x} = 2$$

$$\frac{\partial z}{\partial y} = -6y \qquad \frac{\partial z}{\partial y} = -6$$

vectors in tangent plane

$\langle 1, 0, 2\rangle$

$\langle 0, 1, -6\rangle$

$$n = \begin{vmatrix} i & j & k\\ 1 & 0 & 2\\ 0 & 1 & -6\end{vmatrix} = \langle -2, 6, 1\rangle$$

plane : $\langle -2, 6, 1\rangle \cdot \langle x-1, y-1, z+2\rangle = 0$

$-2(x-1) + 6(y-1) + (z+2) = 0$: $-2x + 6y + z = 2$.

(b) Level surface of $F(x,y,z) = x^2 - 3y^2 - z$.

$\nabla F = \langle 2x, -6y, -1 \rangle$.

$\nabla F\ (1,1,-2) = \langle 2, -6, -1 \rangle$.

plane : $\langle 2,-6,-1\rangle \cdot \langle x-1, y-1, z+2 \rangle = 0$

$2(x-1) - 6(y-1) - (z+2) = 0$

$2x - 6y - z = -2$

12. $\nabla f = \langle 2x+3y,\ 3x-2y \rangle$

$\nabla f(2,3) = \langle 13, 0 \rangle$ $\quad\quad$ $\underline{u} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$D_{\underline{u}} f(2,3) = \nabla f(2,3) \cdot \underline{u} = \frac{13}{\sqrt{5}}$.

unit vector in direction of greatest increase. $\frac{\nabla f}{|\nabla f|} = \langle 1, 0 \rangle$