Math 3400 Test 1. Spring 2014

1. Give a set of parametric equations for the line through the points $(1, 2, 5)$ and $(-3, 1, -1)$.

2. Give the equation of the plane which contains the following two lines:

$$r_1(t) = \langle 1 + t, 1 - t, 2 + 3t \rangle,$$
$$r_2(t) = \langle 3t, 2 + t, -1 + t \rangle.$$

3. Perform the indicated matrix multiplication

$$\begin{pmatrix} 2 & t \\ t & 1 \\ 3 & -t \end{pmatrix} \cdot \begin{pmatrix} t & 1 & 2 \\ 1 & t & -t^2 \end{pmatrix}.$$

4. Sketch the solid described by the following (a) cylindrical and (b) spherical inequalities.

(a) $1 \le r \le 2,\ 0 \le \theta \le \pi/2,\ 0 \le z \le 1$

(b) $0 \le \rho \le 1,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le \pi/2.$

5. Sketch (in the xy-plane) the domains of each of the following functions.

(a) $f(x, y) = \sqrt{xy}$

(b) $f(x, y) = \sqrt{x - y}$

(c) $f(x, y) = \dfrac{1}{(x - 1)(y - 2)}.$

6. Demonstrate that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 + 2y^2}.$$

7. Compute the partial derivative $\partial f/\partial x$ and $\partial f/\partial y$ of the following function

$$f(x, y) = e^{xy} \sin x.$$

8. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the function given by the equation

$$f(s, t) = (s^2 - t^2, 2st).$$

Compute Df.

9. Give the equation of the tangent plane to the surface $x^2 + y^3 + z = 1$ at the point $(1, 1, -1)$.

10. Given a function $f(x, y, z)$, give a "chain rule" to compute $df/d\phi$ where

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

1.

$$\begin{aligned} r_0 &= \langle 1, 2, 5 \rangle \\ d &= \langle -4, -1, -6 \rangle \end{aligned}$$

$$\begin{aligned} r(t) &= r_0 + td \\ &= \langle 1, 2, 5 \rangle + t\langle -4, -1, -6 \rangle \\ &= \langle 1 - 4t, 2 - t, 5 - 6t \rangle \end{aligned}$$

2. The direction vectors of the two lines are vectors in the plane: $v_1 = \langle 1, -1, 3 \rangle$ and $v_2 = \langle 3, 1, 1 \rangle$. The normal vector to the plane is

$$n = v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \langle -4, 8, 4 \rangle.$$

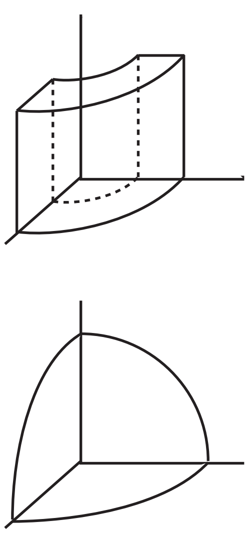
The equation of the plane is then

$$\begin{aligned} \langle -4, 8, 4 \rangle \cdot \langle x - 1, y - 1, z - 2 \rangle &= 0 \\ -4(x-1) + 8(y-1) + 4(z-2) &= 0 \\ -4x + 4 + 8y - 8 + 4z - 8 &= 0 \\ -4x + 8y + 4z &= 12 \\ -x + 2y + z &= 3 \end{aligned}$$

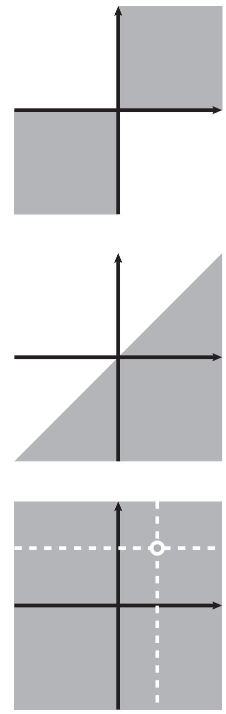
3.

$$= \begin{bmatrix} 3t & 2 + t^2 & 4 - t^3 \\ t^2 + 1 & 2t & 2t - t^2 \\ 2t & 3 - t^2 & 6 + t^3 \end{bmatrix}$$

4.



5. The domains are the shaded gray regions.



6. Consider the limit along two different paths approaching zero: along the path $\langle t, 0 \rangle$,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 + 2y^2} = \lim_{t\to 0} \frac{t^2 + 0^2}{t^2 + 0^2} = 1.$$

Along the path $\langle 0, t \rangle$,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 + 2y^2} = \lim_{t\to 0} \frac{0^2 + t^2}{0^2 + 2t^2} = \frac{1}{2}$$

Since the limits along these two paths are not equal, the limit does not exist.

7.

$$\frac{\partial f}{\partial x} = e^{xy} \cos x + y e^{xy} \sin x$$

$$\frac{\partial f}{\partial y} = x e^{xy} \sin x$$

8.

$$Df = \begin{bmatrix} 2s & -2t \\ 2t & 2s \end{bmatrix}$$

9. This is a level surface of the function $f(x, y, z) = x^2 + y^3 + z$. The gradient of this function is $\nabla f = \langle 2x, 3y^2, 1\rangle$, and in particular,

$$\nabla f(1, 1, -1) = \langle 2, 3, 1\rangle.$$

The equation of the plane is then

$$\begin{aligned}\langle 2, 3, 1\rangle \cdot \langle x - 1, y - 1, z + 1\rangle &= 0 \\ 2(x - 1) + 3(y - 1) + (z + 1) &= 0 \\ 2x + 3y + z &= 4\end{aligned}$$

10.

$$\begin{aligned}\frac{\partial f}{\partial \phi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi} \\ &= \frac{\partial f}{\partial x}\rho\cos\phi\cos\theta \\ &\qquad + \frac{\partial f}{\partial y}\rho\cos\phi\sin\theta \\ &\qquad\qquad - \frac{\partial f}{\partial z}\rho\sin\phi.\end{aligned}$$