The Precise Definition of the Limit

$\lim_{x \to a} f(x) = L$ if:
For every $\epsilon > 0$ there exists a $\delta > 0$ so that if

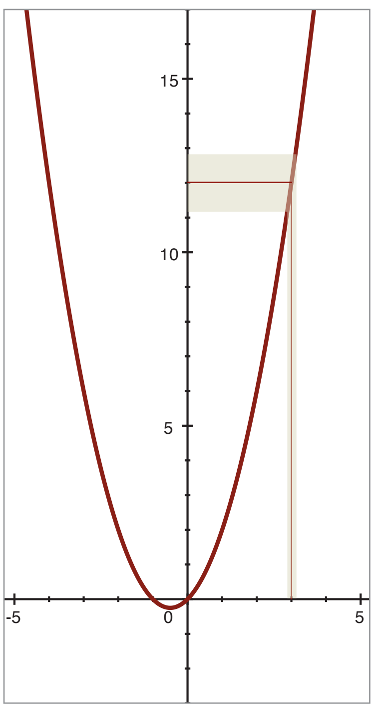
$$|x - a| < \delta$$

then

$$|f(x) - L| < \epsilon.$$

Example: give a delta-epsilon proof that:

$$\lim_{x \to 3} x^2 + x = 12$$



Fix $\epsilon > 0$.

Goal: find $\delta > 0$ so that if $|x - 3| < \delta$, then

$$|(x^2 + x) - 12| < \epsilon.$$

Work backwards:

$$|x^2 + x - 12| < \epsilon$$
$$|(x + 4)(x - 3)| < \epsilon$$
$$|x + 4| \cdot |x - 3| < \epsilon$$
$$|x - 3| < \frac{\epsilon}{|x + 4|}$$

Restrict δ to be at most 1. In this case, $2 \leq x \leq 4$, and so

$$\frac{\epsilon}{8} \leq \frac{\epsilon}{x + 4} \leq \frac{\epsilon}{6}.$$

Based on these calculations, we should set

$$\delta = \min\{1, \epsilon/8\}.$$

Verification: If $|x - 3| < \min\{1, \epsilon/8\}$, then $2 \leq x \leq 4$ and so

$$\begin{aligned} |(x^2 + x) - 12| &= |(x + 4)(x - 3)| \\ &= |x + 4||x - 3| \\ &< 8 \cdot \epsilon/8 \\ &= \epsilon. \end{aligned}$$