

**0. YOU CANT DEFINE EVERYTHING  
AXIOMS AND MODELS**



## Axiom

Let's start with some very basic things. This book is about plane geometry, and in plane geometry you can't get much more basic than points and lines. So let's start there. The first thing to realize is that both of these things, points and lines, are abstractions. You will not find them in the real world. Oh sure, there are point-like things out there— atoms might be a good example. There are line-like things too— laser beams come to mind. But these physical manifestations fall short of “true” points and lines. Points and lines, in other words, are not things we can point to in the real world. In a casual setting, that may not be a big deal. After all, the whole of human experience requires us to deal with abstraction in a variety of contexts on a daily basis. But to try to develop a precise mathematical system from these abstractions— well, that is a little bit more problematic. Consider the opening statements in Euclid's *Elements*,

Definition 1. A point is that which has no part.

Definition 2. A line is breadthless length.

I have to admit, I do like those definitions. They are kind of poetic (at least as poetic as mathematics is permitted to be). But let's be honest— how much information do they really convey? Euclid doesn't define a part, nor does he define breadth or length. Were he to define those terms, they would have to be described using other terms, which would in turn need their own definition, and so on. It isn't that Euclid's definitions are bad. It is that this is a hopeless situation. You can't define everything.

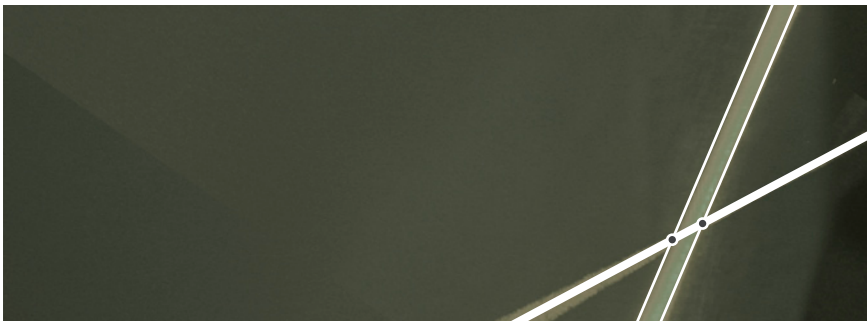
Modern geometry takes an entirely different approach to the issue of elementary definitions. In truth, I think it would be fair to say that modern geometry dodges the question. But it does so in such an artful way that you almost feel foolish for asking the question in the first place. Like its classical counterpart, modern geometry is built upon a foundation of a few basic terms, such as point and line. Unlike the classical approach, in modern geometry no effort is made to define those basic terms. In fact, they are called the *undefineds* of the system. Well, you may ask, what can I do with terms that have no meaning? This is where the *axioms* of the geometry come into play. All the behavior, properties and interactions of the undefined terms are described in a set of statements called the *axioms* of the system. No effort is made to argue for the truth of the axioms. How could you do so?— they are statements about terms which themselves have no meaning. As long as the axioms do not contradict one another, they will define some kind of geometry. It may be quite different from

the Euclidean geometry to which we are accustomed, but it is a geometry none the less.

## Model

Okay, you say, I see what you are saying, but I have done geometry before, and I really like those pictures and diagrams. They help me to understand what is going on. Well, I agree completely! Sure, a bad diagram can be misleading. Even a good diagram can be misleading at times. On the whole, though, I believe that diagrams lead more often than they mislead. The very thesis of this book is that illustrations are an essential part of the subject.

In that case, what is the relationship between illustrations and axioms? First of all, we have to accept that the illustrations are imperfect. Lines printed on paper have a thickness to them. They are finite in length. Points also have a length and width— otherwise we couldn't see them. That's just the way it has to be. But really, I don't think that is such a big deal. I think the focus on those imperfections tends to mask an even more important issue. And that is that these illustrations represent only one manifestation of the axioms. Points and lines as we depict them are one way to interpret the undefined terms of point and line. This interpretation happens to be consistent with all of the standard Euclidean axioms. But there may be a completely different interpretation of the undefineds which also satisfies the Euclidean axioms. Any such interpretation is called a *model* for the geometry. A geometry may have many models, and from a theoretical point of view, no one model is more right than any other. It is important, then, to prove facts about the geometry itself, and not peculiarities of one particular model.



## Fano's Geometry

To see how axiomatic geometry works without having our Euclidean intuition getting in the way, let's consider a decidedly non-Euclidean geometry called Fano's geometry (named after the Italian algebraic geometer Gino Fano). In Fano's geometry there are three undefined terms, *point*, *line*, and *on*. Five axioms govern these undefined terms.

*Ax 1.* There exists at least one line.

*Ax 2.* There are exactly three points on each line.

*Ax 3.* Not all points are on the same line.

*Ax 4.* There is exactly one line on any two distinct points.

*Ax 5.* There is at least one point on any two distinct lines.

Fano's geometry is a simple example of what is called a finite projective geometry. It is projective because, by the fifth axiom, all lines intersect one another (lines cannot be parallel). It is finite because, as we will see, it only contains finitely many points and lines. To get a sense of how an axiomatic proof works, let's count the points and lines in Fano's geometry.

THM

Fano's geometry has exactly seven points and seven lines.

*Proof.* I have written this proof in the style I was taught in high school geometry, with a clear separation of each statement and its justification (in this case, an axiom). It is my understanding that geometry is rarely taught this way now. A shame, I think, since I think that this is a good way to introduce the idea of logical thought and proof.

LN PT	1	2	3	4
1	○	○		
2	○		○	
3	○			○
4		○	○	○

*Ax 1* There is a line  $\ell_1$ .

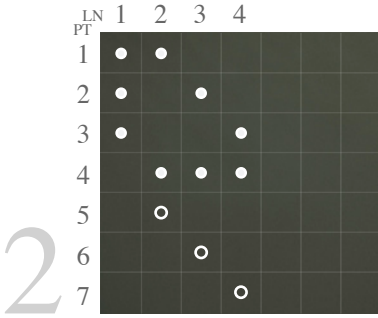
*Ax 2* On  $\ell_1$ , there are three points. Label them  $p_1$ ,  $p_2$  and  $p_3$ .

*Ax 3* There is a fourth point  $p_4$  that is not on  $\ell_1$ .

*Ax 4* There are lines:  $\ell_2$  on  $p_1$  and  $p_4$ ,  $\ell_3$  on  $p_2$  and  $p_4$ , and  $\ell_4$  on  $p_3$  and  $p_4$ . Each of these lines is distinct.

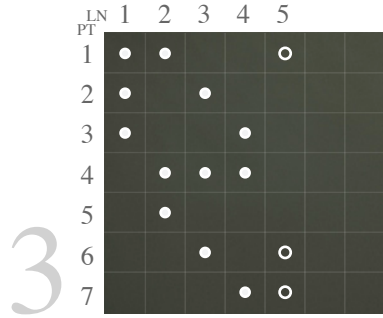
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This chart tracks the incidences of points on lines through the proof.



Ax 2 Each of these lines has a third point on it.

Ax 4 They are distinct and different from any of the previously declared points. Label them:  $p_5$  on  $\ell_2$ ,  $p_6$  on  $\ell_3$ , and  $p_7$  on  $\ell_4$ .

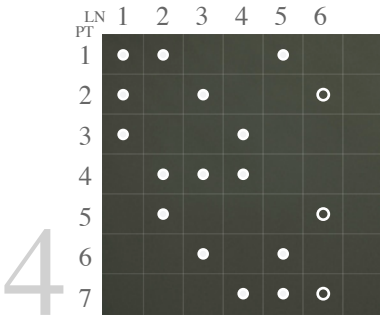


Ax 4 There must be a line  $\ell_5$  on  $p_1$  and  $p_6$ .

Ax 2 The line  $\ell_5$  must have one more point on it.

Ax 4 That point cannot be either  $p_3$  or  $p_4$ .

Ax 5 For  $\ell_5$  and  $\ell_4$  to intersect, the third point of  $\ell_5$  must be  $p_7$ .

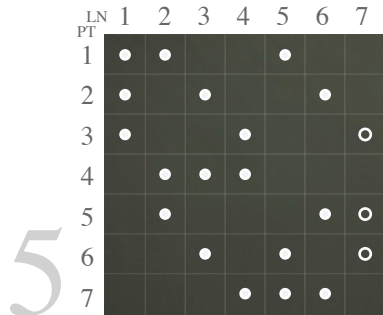


Ax 4 There must be a line  $\ell_6$  on  $p_2$  and  $p_5$ .

Ax 2 The line  $\ell_6$  must have a third point on it.

Ax 4 That point cannot be  $p_3$  or  $p_4$ .

Ax 5 For  $\ell_6$  and  $\ell_4$  to intersect, the third point of  $\ell_6$  must be  $p_7$ .



Ax 4 There must be a line  $\ell_7$  on  $p_3$  and  $p_5$ .

Ax 2 The line  $\ell_7$  must have one more point on it.

Ax 4 That point cannot be  $p_2$  or  $p_4$ .

Ax 5 For  $\ell_7$  and  $\ell_3$  to intersect, the third point of  $\ell_7$  must be  $p_6$ .

We now have seven points and seven lines as required. Could there be more? Let's suppose there were an eighth point  $p_8$ .

*Ax 4* Then there would be a line  $\ell_8$  on  $p_1$  and  $p_8$ .

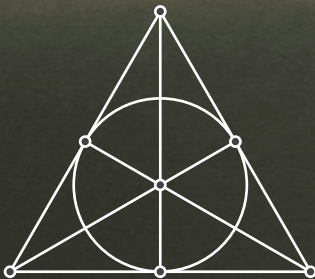
*Ax 3* Line  $\ell_8$  would have to have another point on it.

*Ax 4* This other point would have to be distinct from each of  $p_2$  through  $p_7$ .

*Ax 5* Then  $\ell_8$  would not share a point with  $\ell_3$  (and other lines as well). Thus there cannot be an eighth point.

*Ax 4* There is now a line on every pair of points. Therefore there can be no more lines.

□



**A model for Fano's geometry.**

The nodes of the graph represent the points. The six segments and the circle represent the lines.

## Further reading

Euclid's *Elements* is still a fantastic read. There are several editions available, both in text form and online, including, for instance, [3]. If you want to know more about projective geometry in general, I would recommend Coxeter's book [2]. For a finite projective planes, I have found a nice set of online notes by Jürgen Bierbrauer [1]. At the time of this writing they are available at the web address:

<http://www.math.mtu.edu/~jbierbra/HOMEZEUGS/finitegeom04.ps>.

- [1] Jürgen Bierbrauer. Finite geometries: MA5980. Lecture notes distributed on World Wide Web, 2004.
- [2] H.S.M. Coxeter. *Projective Geometry*. Blaisdell Publishing Co., New York, 1st edition, 1964.
- [3] Euclid. *The Thirteen Books of Euclid's Elements*. Dover Publications, New York, 2nd edition, 1956. Translated from the text of Heiberg, with introduction and commentary by Sir Thomas L. Heath.

