LESSON 10

TRIANGLES IN NEUTRAL GEOMETRY

THREE THEOREMS OF MEASUREMENT

In this lesson we are going to take our newly created measurement systems, our rulers and our protractors, and see what we can tell us about triangles. We will derive three of the most fundamental results of neutral geometry: the Saccheri-Legendre Theorem, the Scalene Triangle Theorem, and the Triangle Inequality.

The Saccheri-Legendre Theorem

The Saccheri-Legendre Theorem is a theorem about the measures of the interior angles of a triangle. For the duration of this lesson, if $\triangle ABC$ is any triangle, I will call

$$s(\triangle ABC) = (\angle A) + (\angle B) + (\angle C)$$

the *angle sum* of the triangle. As you probably know, in Euclidean geometry the angle sum of any triangle is 180°. That is not necessarily the case in neutral geometry, though, so we will have to be content with a less restrictive (and less useful) condition.

THE SACCHERI LEGENDRE THEOREM
For any triangle $\triangle ABC$, $s(\triangle ABC) \leq 180^\circ$.

I will prove this result in three parts– two preparatory lemmas followed by the proof of the main theorem.

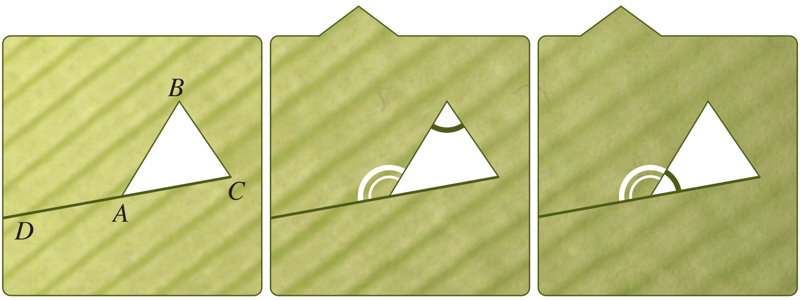
Euclidean triangle *non-Euclidean triangle* *not a neutral triangle*

LEMMA ONE

The sum of the measures of any two angles in a triangle is less than 180°.

Proof. Let's suppose that we are given a triangle $\triangle ABC$ and we want to show that $(\angle A) + (\angle B) < 180°$. First I need to label one more point: choose D so that $D * A * C$. Then

$$(\angle BAC) + (\angle ABC) < (\angle BAC) + (\angle BAD) = 180°.$$



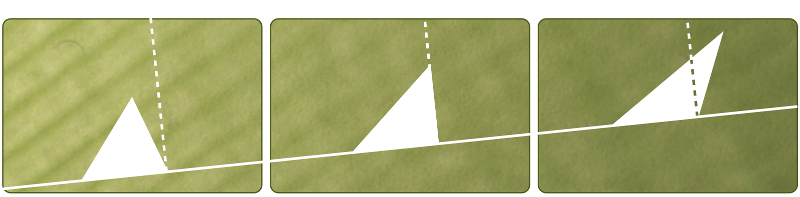
Exterior Angle Th. $(\angle ABC) < (\angle BAD)$ *Supplements* $\angle BAC$ and $\angle BAD$

□

Note that this means that a triangle cannot support more than one right or obtuse angle– if a triangle has a right angle, or an obtuse angle, then the other two angles have to be acute. That leads to some more terminology.

DEF: ACUTE, RIGHT, AND OBTUSE TRIANGLES

A triangle is *acute* if all three of its angles are acute. A triangle is *right* if it has a right angle. A triangle is *obtuse* if it has an obtuse angle.



Acute *Right* *Obtuse*

The real key to this proof of the Saccheri-Legendre Theorem, the mechanism that makes it work, is the second lemma.

LEMMA TWO

For any triangle $\triangle ABC$, there is another triangle $\triangle A'B'C'$ so that

1. $s(\triangle ABC) = s(\triangle A'B'C')$, and
2. $(\angle A') \leq (\angle A)/2$.

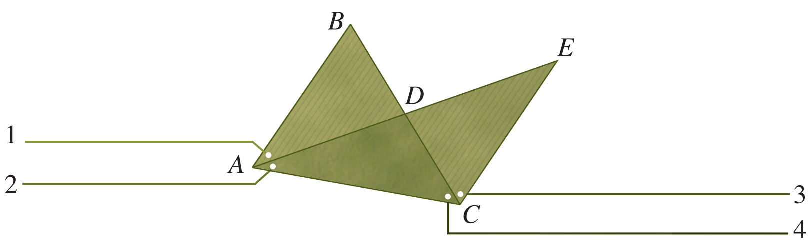
Proof. This is a constructive proof: I am going to describe how to build a triangle from $\triangle ABC$ that meets both of the requirements listed in the theorem. First we are going to need to label a few more points:

D: the midpoint of BC,

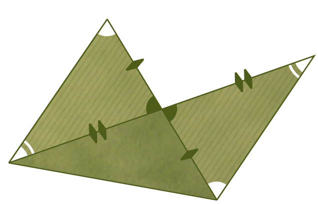
E: on $AD\rightarrow$, so that $A * D * E$ and $AD \simeq DE$.

My claim is that $\triangle ACE$ satisfies both of the conditions (1) and (2). Showing that it does involves comparing angle measures, and with that in mind I think it is helpful to abbreviate some of the angles:

$\angle 1$ for $\angle BAD$, $\angle 2$ for $\angle DAC$, $\angle 3$ for $\angle DCE$, and $\angle 4$ for $\angle ACD$.



The key to showing that $\triangle ACE$ meets requirements (1) and (2) is the pair of congruent triangles formed by carving away the overlap of $\triangle ABC$ and $\triangle ACE$. Notice that by S·A·S



$BD \simeq CD$

$\angle BDA \simeq \angle CDE$ $\quad\quad\quad\quad$ $\angle 1 \simeq \angle E$

$DA \simeq DE \longrightarrow \triangle BDA \simeq \triangle CDE \longrightarrow \angle B \simeq \angle 3$

Condition 1. For the first, all you have to do is compare the two angle sums:

$$s(\triangle ABC) = (\angle A) + (\angle B) + (\angle 4) = (\angle 1) + (\angle 2) + (\angle B) + (\angle 4)$$

$$s(\triangle ACE) = (\angle 2) + (\angle ACE) + (\angle E) = (\angle 2) + (\angle 3) + (\angle 4) + (\angle E).$$

Sure enough, they are the same.

Condition 2. The second part is a little devious, because I can't tell you which angle of $\triangle ACE$ will end up being $\angle A'$. What I can say, though, is that

$$(\angle BAC) = (\angle 1) + (\angle 2) = (\angle E) + (\angle 2).$$

Therefore it isn't possible for both $\angle E$ and $\angle 2$ to measure more than $(\angle BAC)/2$. Let $\angle A'$ be the smaller of the two (or just choose one if they are both the same size). $\square$

Now we can combine those two lemmas into a proof of the Saccheri-Legendre Theorem itself.

Proof. Suppose that there is a triangle $\triangle ABC$ whose angle sum is more than $180°$. In order to keep track of that excess, write

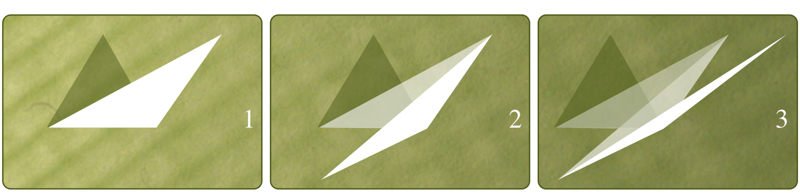
$$s(\triangle ABC) = (180 + x)°.$$

Now let's iterate! According to Lemma 2, there is a triangle

$\triangle A_1B_1C_1$ with the same angle sum but $(\angle A_1) \le \frac{1}{2}(\angle A)$;

$\triangle A_2B_2C_2$ with the same angle sum but $(\angle A_2) \le \frac{1}{2}(\angle A_1) \le \frac{1}{4}(\angle A)$;

$\triangle A_3B_3C_3$ with the same angle sum but $(\angle A_3) \le \frac{1}{2}(\angle A_2) \le \frac{1}{8}(\angle A)$;



Starting from an equilateral triangle, the first three iterations.

After going through this procedure n times, we will end up with a triangle $\triangle A_nB_nC_n$ whose angle sum is still $(180+x)^\circ$ but with one very tiny angle– $(\angle A_n) \leq \frac{1}{2^n}(\angle A)$. No matter how big $\angle A$ is or how small x is, there is a large enough value of n so that $\frac{1}{2^n}(\angle A) < x$. In that case, the remaining two angles of the triangle $\angle B_n$ and $\angle C_n$ have to add up to more than 180°. According to Lemma 1, this cannot happen. Therefore there cannot be a triangle with an angle sum over 180°. □

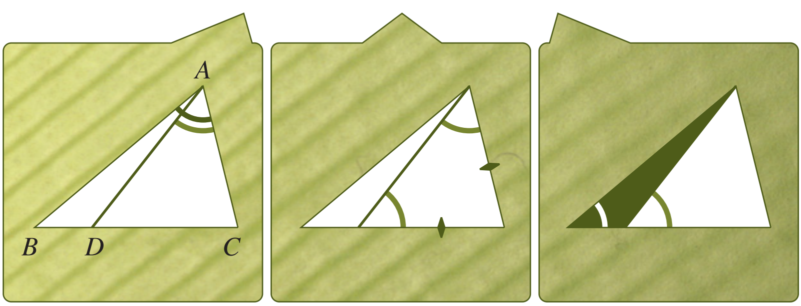
The Scalene Triangle Theorem

The Scalene Triangle Theorem relates the measures of the angles of triangle to the measures of its sides. Essentially, it guarantees that the largest angle is opposite the longest side and that the smallest angle is opposite the shortest side. More precisely

THM: SCALENE TRIANGLE THEOREM
In $\triangle ABC$ suppose that $|BC| > |AC|$. Then $(\angle BAC) > (\angle ABC)$.

Proof. With the results we have established so far, this is an easy one. We need to draw an isosceles triangle into $\triangle ABC$ and that requires one additional point. Since $|BC| > |AC|$, there is a point D between B and C so that $CA \simeq CD$. Then

$$(\angle BAC) > (\angle DAC) = (\angle ADC) > (\angle ABC).$$



D is in the interior of $\angle BAC$. *Isosceles Triangle Th.* *Exterior Angle Th.*

□

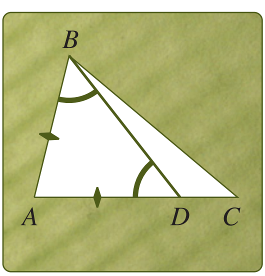
The Triangle Inequality

The Triangle Inequality deals with the lengths of the three sides of a triangle, providing upper and lower bounds for one side in terms of the other two. This is one of the results that has escaped the confines of neutral geometry, though, and you will see triangle inequalities in various disguises is many different areas of math.

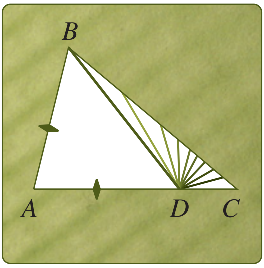
THM: THE TRIANGLE INEQUALITY
In any triangle $\triangle ABC$, the length of side AC is bounded above and below by the lengths of AB and BC:

$$||AB| - |BC|| < |AC| < |AB| + |BC|.$$

Proof. The second inequality is usually what people think of when they think of the Triangle Inequality, and that's the one that I am going to prove. I will leave the proof of the first inequality to you. The second inequality is obviously true if AC isn't the longest side of the triangle, so let's focus our attention on the only really interesting case– when AC is the longest side. As in the proof of the Scalene Triangle Theorem, we are going to build an isosceles triangle inside $\triangle ABC$. To do that, label D between A and C so that $AD \simeq AB$. According to the Isosceles Triangle Theorem, $\angle ADB \simeq \angle ABD$. Thanks to the Saccheri-Legendre Theorem, we now know that these angles can't both be right or obtuse, so they have to be acute. Therefore, $\angle BDC$, which is supplementary to $\angle ADB$, is obtuse. Again, the Saccheri-Legendre Theorem: the triangle $\triangle BDC$ will



In $\triangle ABD$, $\angle B$ and $\angle D$ are congruent, so they must be acute.



In $\triangle BCD$, $\angle D$ is obtuse, so it is the largest angle. Opposite it, BC is the longest side.

only support one obtuse angle, so $\angle BDC$ has to be the largest angle in that triangle. According to the Scalene Triangle Theorem, BC has to be the longest side of $\triangle BDC$. Hence $|DC| < |BC|$. Now let's put it together

$$|AC| = |AD| + |DC| < |AB| + |BC|.$$

$\square$

For proper triangles, the Triangle Inequality promises strict inequalities– $<$ instead of $\leq$. When the three points A, B and C collapse into a straight line, they no longer form a proper triangle, and that is when the inequalities become equalities:

$$\begin{aligned}
&\text{if } C * A * B, \text{ then } |AC| = |BC| - |AB|;\\
&\text{if } A * C * B, \text{ then } |AC| = |AB| - |BC|;\\
&\text{if } A * B * C, \text{ then } |AC| = |AB| + |BC|.
\end{aligned}$$

Exercises

1. Prove the converse of the Scalene Triangle Theorem: in $\triangle ABC$, if $(\angle BAC) > (\angle ABC)$ then $|BC| > |AC|$.

2. Prove the other half of the triangle inequality.

3. Given a triangle $\triangle ABC$, consider the interior and exterior angles at a vertex, say vertex A. Prove that the bisectors of those two angles are perpendicular.

4. Prove that for any point P and line ℓ, there are points on ℓ which are arbitrarily far away from ℓ.

5. Prove that equilateral triangles exist in neutral geometry (that is, describe a construction that will yield an equilateral triangle). Note that all the interior angles of an equilateral triangle will be congruent, but you don't know that the measures of those interior angles is 60°.

6. Prove a strengthened form of the Exterior Angle Theorem: for any triangle, the measure of an exterior angle is greater than or equal to the *sum* of the measures of the two nonadjacent interior angles.

7. Prove that if a triangle is acute, then the line which passes through a vertex and is perpendicular to the opposite side will intersect that side (the segment, that is, not just the line containing the segment).

Recall that SSA is not a valid triangle congruence theorem. If you know just a little bit more about the triangles in question, though, SSA can be enough to prove triangles congruent. The next questions look at some of those situations.

8. In a right triangle, the side opposite the right angle is called the *hypotenuse*. By the Scalene Triangle Theorem, it is the longest side of the triangle. The other two sides are called the *legs* of the triangle. Consider two right triangles $\triangle ABC$ and $\triangle A'B'C'$ with right angles at C and C', respectively. Suppose in addition that

$$AB \simeq A'B' \quad \& \quad AC \simeq A'C'$$

(the hypotenuses are congruent, as are one set of legs). Prove that $\triangle ABC \simeq \triangle A'B'C'$. This is the H·L congruence theorem for right triangles.

9. Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are *acute* triangles and that

$$AB \simeq A'B' \quad BC \simeq B'C' \quad \angle C \simeq \angle C'.$$

Prove that $\triangle ABC \simeq \triangle A'B'C'$.

10. Consider triangles $\triangle ABC$ and $\triangle A'B'C'$ with

$$AB \simeq A'B' \quad BC \simeq B'C' \quad \angle C \simeq \angle C'.$$

Suppose further that $|AB| > |BC|$. Prove that $\triangle ABC \simeq \triangle A'B'C'$.

References

The proof that I give for the Saccheri-Legendre Theorem is the one I learned from Wallace and West's book [1].

[1] Edward C. Wallace and Stephen F. West. *Roads to Geometry*. Pearson Education, Inc., Upper Saddle River, New Jersey, 3rd edition, 2004.