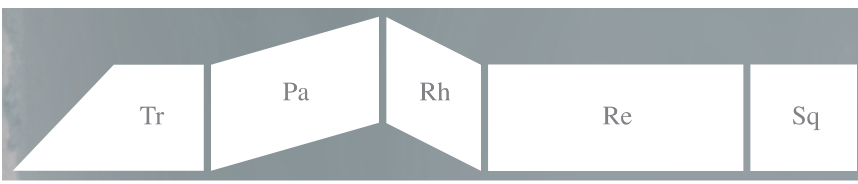
12 FIVE EASY PIECES

QUADRILATERAL CONGRUENCE THEOREMS

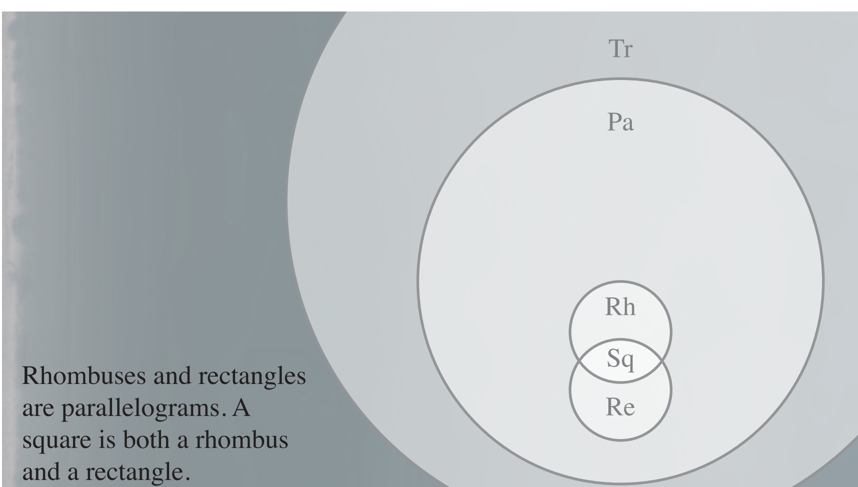
This is the last lesson in neutral geometry. After this, we will allow ourselves one more axiom dealing with parallel lines, and that is the axiom which turns neutral geometry into Euclidean geometry. Before turning down the Euclidean path, let's spend just a little time looking at quadrilaterals. The primary goal of this section will be to develop quadrilateral congruence theorems similar to the triangle congruence theorems we picked up in earlier lessons.

Terminology

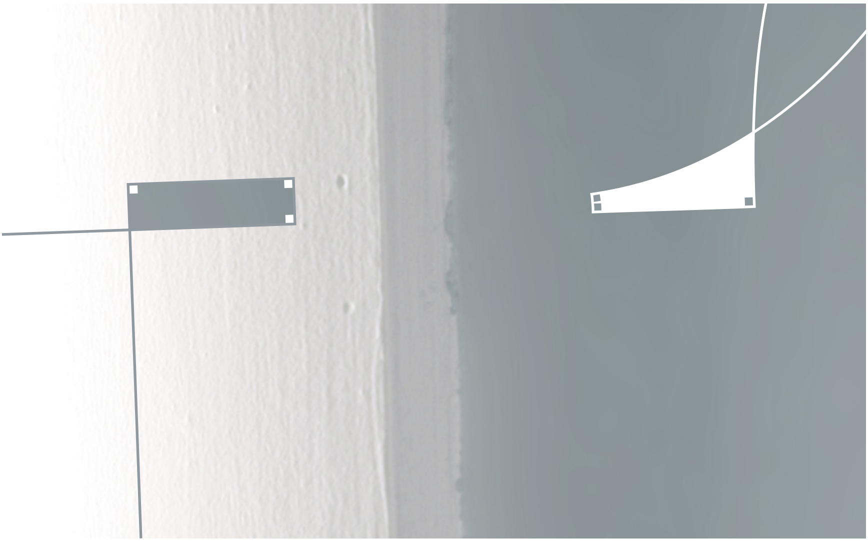
Before I start working on congruence theorems, though, let me quickly run through the definitions of a few particular types of quadrilaterals.



Trapezoid	a pair of parallel sides
Parallelogram	two pairs of parallel sides
Rhombus	four congruent sides
Rectangle	four right angles
Square	four congruent sides and four right angles



Rhombuses and rectangles are parallelograms. A square is both a rhombus and a rectangle.



Quadrilaterals with three right angles. On the left, in Euclidean geometry, the fourth angle is a right angle. On the right, in non-Euclidean geometry, the fourth angle is acute.

One of the risks that you run when you define an object by requiring it to have certain properties, as I have done above, is that you may define something that cannot be– something like an equation with no solution. The objects I have defined above are all such common shapes in everyday life that we usually don't question their existence. Here's the interesting thing though– in neutral geometry, there is no construction which guarantees you can make a quadrilateral with four right angles– that is, neutral geometry does not guarantee the existence of rectangles or squares. At the same time, it does nothing to prohibit the existence of squares or rectangles either. You can make a quadrilateral with three right angles pretty easily, but once you have done that, you have no control over the fourth angle, and the axioms of neutral geometry are just not sufficient to prove definitively whether or not that fourth angle is a right angle. This is one of the fundamental differences that separates Euclidean geometry from non-Euclidean geometry. In Euclidean geometry, the fourth angle is a right angle, so there are rectangles. In non-Euclidean geometry, the fourth angle cannot be a right angle, so there are no rectangles. When we eventually turn our attention to non-Euclidean geometry, I want to come back to this– I would like to begin that study with a more thorough investigation of these quadrilaterals that try to be like rectangles, but fail.

Quadrilateral Congruence

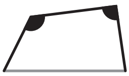
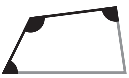
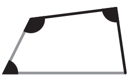
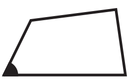
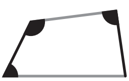
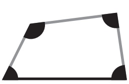
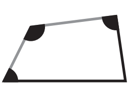
I feel that many authors view the quadrilateral congruences as a means to an end, and as such, tend to take a somewhat ad hoc approach to them. I think I understand this approach– the quadrilateral congruence theorems themselves are a bit bland compared to their application. Still, I want to be a bit more systematic in my presentation of them. In the last chapter we looked at several classes of polygons. To recap:

$$\{\text{convex polygons}\} \subset \{\text{simple polygons}\} \subset \{\text{polygons}\}.$$

For what we are going to be doing in this book, we really only need the congruence results for convex quadrilaterals, but I am going to try to tackle the slightly broader question of congruence for simple quadrilaterals. While the even broader question of congruence for non-simple quadrilaterals would be interesting, I think it is just too far of a detour.

By definition, two quadrilaterals are congruent if four corresponding sides and four corresponding interior angles are congruent– that's a total of eight congruences. Each congruence theorem says that you can guarantee congruence with some subset of that list. If you recall, for triangles you generally needed to know three of the six pieces of information. For quadrilaterals, it seems that the magic number is five. So what I would like to do in this lesson is to look at all the different possible combinations of five pieces (sides and angles) of a quadrilateral and determine which lead to valid congruence theorems. I won't give all the proofs or all the counterexamples (that way you can tackle some of them on your own), but I will provide the framework for a complete classification.

The first step is some basic combinatorics. Each of these theorems has a five letter name consisting of some mix of *S*s and *A*s. When forming this name, there are two choices, *S* and *A* for each of the five letters, and so there are a total of $2^5 = 32$ possible names. Two of these, S·S·S·S·S and A·A·A·A·A, don't make any sense in the context of quadrilateral congruences, though, since a quadrilateral doesn't have five sides or five angles. That leaves thirty different words. Now it is important to notice that not all of these words represent fundamentally different information about the quadrilaterals themselves. For instance, S·S·A·S·A and A·S·A·S·S both represent the same information, just listed in reverse order. Similarly, S·S·A·S·S and S·S·S·S·A both represent the same information– four sides and one angle. Once those equivalences are taken into consideration, we are left with ten potential quadrilateral congruences.

	Word	**Variations**	**Valid congruence?**
	S·A·S·A·S		yes
	A·S·A·S·A		yes
	A·A·S·A·S	SASAA	yes
	S·S·S·S·A	SSSAS SSASS SASSS ASSSS	no(*)
	A·S·A·A·S	SAASA	no
	A·S·A·S·S	SSASA	no
	A·S·S·A·S	SASSA	no
	A·A·A·A·S	AAASA AASAA ASAAA SAAAA	no
	S·S·S·A·A	AASSS ASSSA SAASS SSAAS	no
	A·A·A·S·S	SAAAS SSAAA ASSAA AASSA	yes

(*) a valid congruence theorem for *convex* quadrilaterals

Table 1. Quadrilateral congruence theorems.

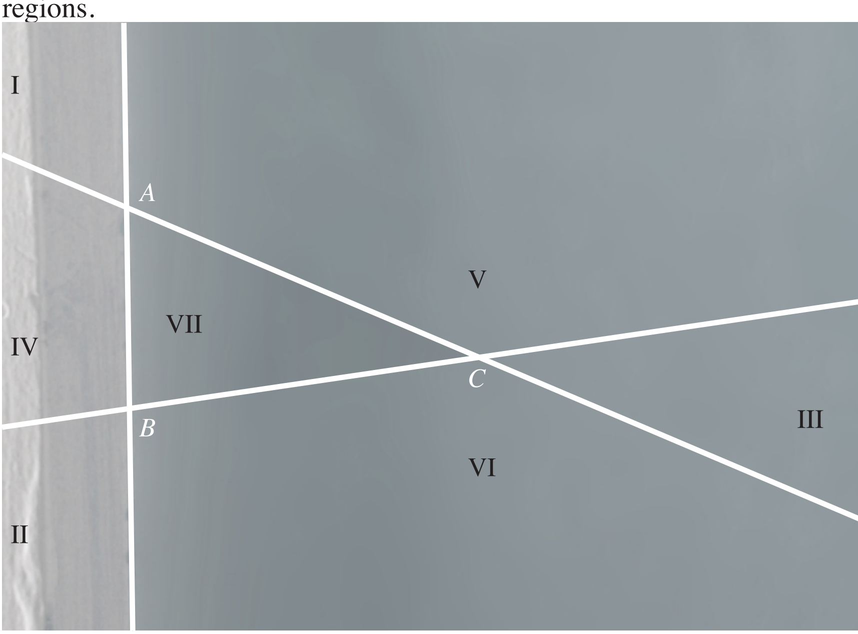
S·A·S·A·S, A·S·A·S·A, and A·A·S·A·S

Each of these is a valid congruence theorem for simple quadrilaterals. The basic strategy for their proofs is to use a diagonal of the quadrilateral to separate it into two triangles, and then to use the triangle congruence theorems. Now the fact that I am allowing both convex and non-convex quadrilaterals in this discussion complicates things a little bit, so let's start by examining the nature of the diagonals of a quadrilateral. Yes, I will be leaving out a few details here (more than a few to be honest) so you should feel free to work out any tricky details for yourself.

Consider a quadrilateral $\square ABCD$ (I am going to use a square symbol to denote a simple quadrilateral). What I want to do is to look at the position of the point D relative to the triangle $\triangle ABC$. Each of the three lines $\leftarrow AB \rightarrow$, $\leftarrow BC \rightarrow$, and $\leftarrow AC \rightarrow$ separate the plane into two pieces. It is not possible, though, for any point of the plane to simultaneously be

(1) on the opposite side of AB from C
(2) on the opposite side of AC from B, and
(3) on the opposite side of BC from A.

Therefore the lines of $\triangle ABC$ divide the plane into seven $(2^3 - 1)$ distinct regions.



The seven "sides" of a triangle.

D is in region	is $\square ABCD$ simple?	D is on the same side of : BC as A	AC as B	AB as C	reflex angle	interior diagonal: AC	BD
I	✓	✓			A	✓	
II	✓		✓		B		✓
III	✓			✓	C	✓	
IV		✓	✓		–	–	–
V	✓	✓		✓	none	✓	✓
VI			✓	✓	–	–	–
VII	✓	✓	✓	✓	D		✓

Table 2. The diagonals of a quadrilateral

Now for each of these seven regions, we can determine whether the diagonals AC and BD are in the interior of $\square ABCD$. Let me point out that this is always an all-or-nothing proposition– either the entire diagonal lies in the interior (excepting of course the endpoints) or none of it does. Additionally, in each case, a diagonal lies in the interior of a quadrilateral if and only if it lies in the interior of both the angles formed by $\square ABCD$ at its endpoints. What I mean is that if, for example, AC is in the interior of $\square ABCD$, then AC will be in the interior of both $\angle DAB$ and $\angle BCD$. If AC isn't in the interior of $\square ABCD$, then AC will not be in the interior of either $\angle DAB$ or $\angle BCD$.

With the diagonals now properly sorted, we can address the congruence theorems directly. Perhaps the most useful of them all is S·A·S·A·S.

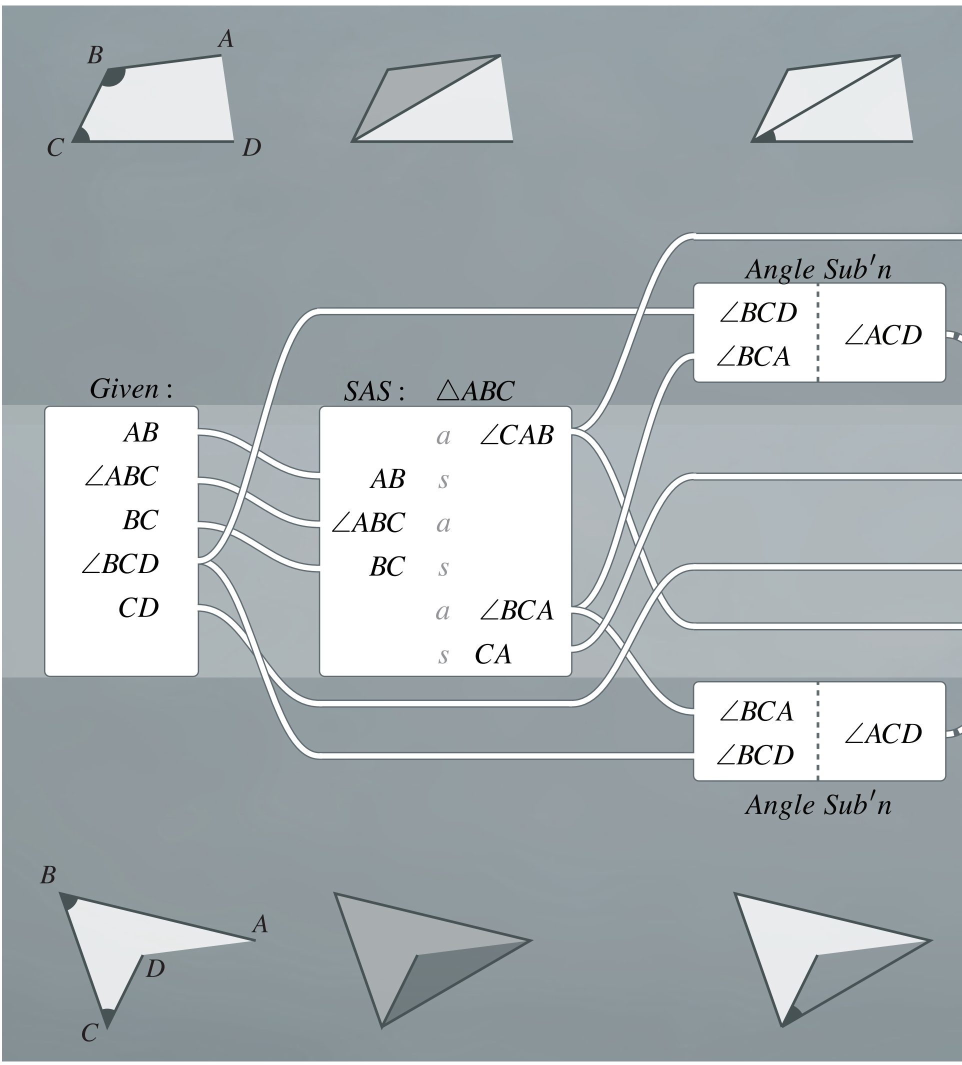
S·A·S·A·S QUADRILATERAL CONGRUENCE

If $\square ABCD$ and $\square A'B'C'D'$ are simple quadrilaterals and

$$AB \simeq A'B' \quad \angle B \simeq \angle B' \quad BC \simeq B'C' \quad \angle C \simeq \angle C' \quad CD \simeq C'D'$$

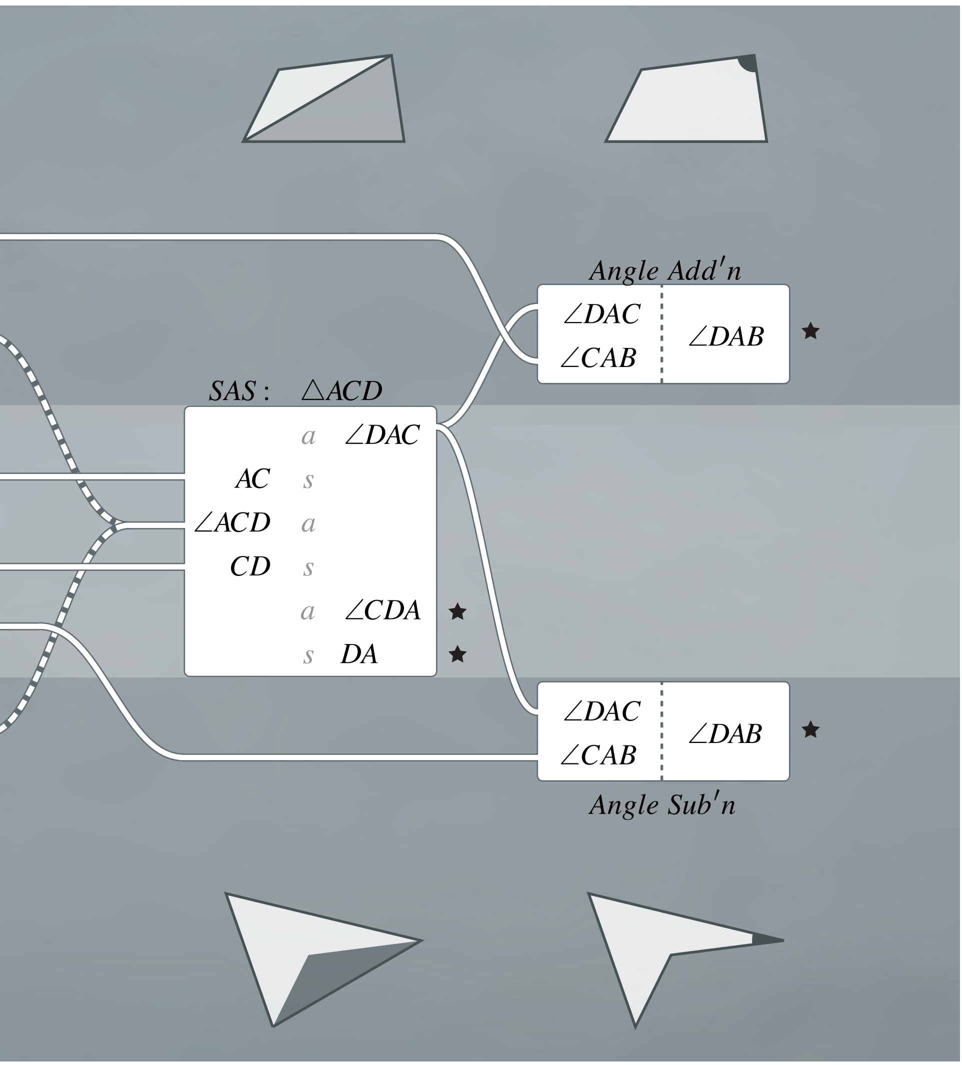
then $\square ABCD \simeq \square A'B'C'D'$.

Proof. The diagonals AC and $A'C'$ are the keys to turning this into a problem of triangle congruence. Unfortunately, we do not know whether or not those diagonals are in the interiors of their respective quadrilaterals. That means we have to tread somewhat carefully at first. Because of S·A·S, $\triangle ABC \simeq \triangle A'B'C'$. You need to pay attention to what is happening at vertex C. If AC is in the interior of the quadrilateral, then it is in the interior of $\angle BCD$ and that means $(\angle BCA) < (\angle BCD)$. Then, since $\angle B'C'A' \simeq \angle BCA$ and $\angle B'C'D' \simeq \angle BCD$, $(\angle B'C'A') < (\angle B'C'D')$. Therefore $A'C'$ must be in the interior of $\angle B'C'D'$ and in the interior of $\square A'B'C'D'$. With the same



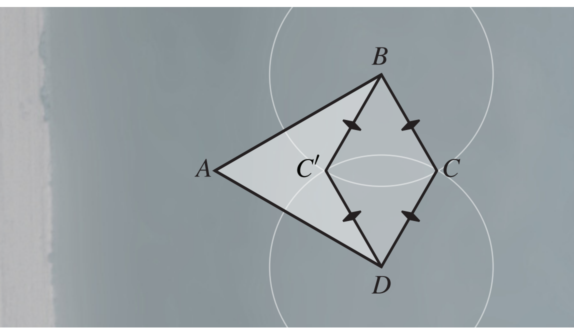
reasoning, we can argue that if AC is not in the interior of $\square ABCD$, then $A'C'$ cannot be in the interior of $\square A'B'C'D'$. So there are two cases, and the assembly of the quadrilateral from the triangles depends upon the case. My diagram of the chase through the congruences is below. I have split it, when necessary, to address the differences in the two cases. $\square$

Using essentially this same approach, you should be able to verify both the A·S·A·S·A and A·A·S·A·S quadrilateral congruences.



S·S·S·S·A

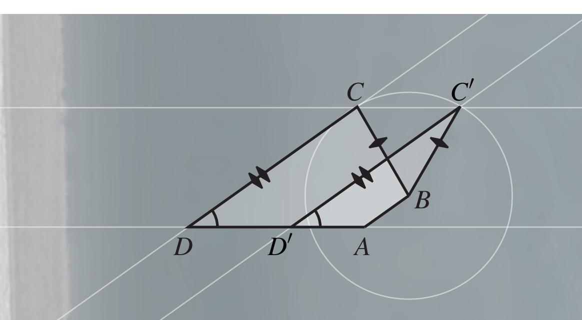
The S·S·S·S·A condition is almost enough to guarantee quadrilateral congruence. Suppose that you know the lengths of all four sides of □*ABCD*, and you also know $\angle A$. Then $\triangle BAD$ is completely determined (S·A·S) and from that $\triangle BCD$ is completely determined (S·S·S). That still does not mean that □*ABCD* is completely determined, though, because there are potentially two ways to assemble $\triangle BAD$ and $\triangle BCD$ (as illustrated). One assembly creates a convex quadrilateral, the other a non-convex one. Now, there will be times when you know the quadrilaterals in question are all convex, and in those situations, S·S·S·S·A can be used to show that convex quadrilaterals are congruent.



Non-congruent quadrilaterals with matching SSSSA. One is convex; the other is not.

A·S·A·A·S, A·S·A·S·S, A·S·S·A·S, A·A·A·A·S, and S·S·S·A·A

None of these provide sufficient information to guarantee congruence and counterexamples can be found in Euclidean geometry. I will just do one of them– S·S·S·A·A, and leave the rest for you to puzzle out. In the illustration below □*ABCD* and □$ABC'D'$ have correponding S·S·S·A·A but are not congruent.



Non-congruent quadrilaterals with matching SSSAA.

A·A·A·S·S

This is the intriguing one. The idea of splitting the quadrilateral into triangles along the diagonal just doesn't work. You fail to get enough information about either triangle. Yet, (as we will see) in Euclidean geometry, the angle sum of a quadrilateral has to be 360°. Since three of the angles are given, that means that in the Euclidean realm the fourth angle is determined as well. In that case, this set of congruences is essentially equivalent to the A·S·A·S·A (which is a valid congruence theorem). The problem is that in neutral geometry the angle sum of a quadrilateral does not have to be 360°. Because of the Saccheri-Legendre Theorem, the angle sum of a quadrilateral cannot be more than 360°, but that is all we can say. It turns out that this *is* a valid congruence theorem in neutral geometry. The proof is a little difficult though. The argument that I want to use requires us to "drop a perpendicular". I have described this process in some of the previous exercises, but let me reiterate here.

LEM 1
For any line ℓ and point P not on ℓ, there is a unique line through P which is perpendicular to ℓ.

The intersection of ℓ and the perpendicular line is often called the *foot* of the perpendicular. The process of finding this foot is called *dropping a perpendicular*. I have already proven the existence part of this– the phrasing was a little different then, but my proof of the existence of right angles (in the lesson on angle comparison) constructs this perpendicular line. As for uniqueness part, I will leave that to you.

LEM 2
Let ℓ be a line, P a point not on ℓ, and Q the foot of the perpendicular to ℓ through P. Then P is closer to Q than it is to any other point on ℓ.

Again, I am going to pass off the proof to you. I would suggest, though, that you think about the Scalene Triangle Theorem. Now on to the main theorem.

A·A·A·S·S QUADRILATERAL CONGRUENCE
If $\square ABCD$ and $\square A'B'C'D'$ are simple quadrilaterals, and

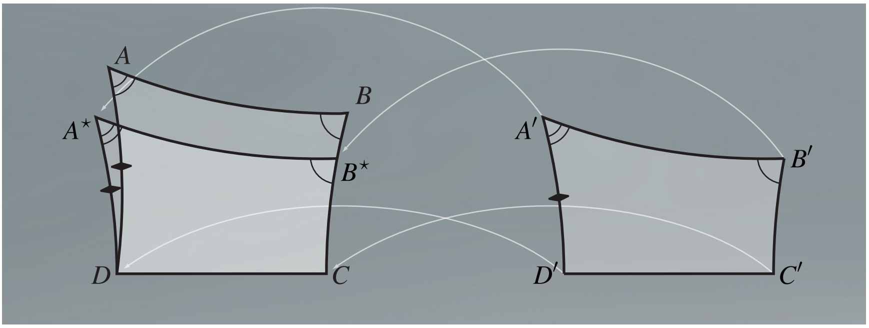
$$\angle A \simeq \angle A' \quad \angle B \simeq \angle B' \quad \angle C \simeq \angle C' \quad CD \simeq C'D' \quad DA \simeq D'A'$$

then $\square ABCD \simeq \square A'B'C'D'$.

Proof. I will use a proof by contradiction of this somewhat tricky theorem. Suppose that $\square ABCD$ and $\square A'B'C'D'$ have the corresponding congruent pieces as described in the statement of the theorem, but suppose that $\square ABCD$ and $\square A'B'C'D'$ are not themselves congruent.

Part One, in which we establish parallel lines.
I want to construct a new quadrilateral: $\square A^\star B^\star CD$ will overlap $\square ABCD$ as much as possible, but will be congruent to $\square A'B'C'D'$. Here is the construction. Locate $B^\star$ on $CB\rightarrow$ so that $CB^\star \simeq C'B'$. Note that BC and $B'C'$ cannot be congruent– if they were the two quadrilaterals would be congruent by A·A·S·A·S. As a result, in the construction, $B \neq B^\star$. The other point to place is $A^\star$. It needs to be positioned so that:

1. it is on the same side of $\leftarrow BC \rightarrow$ as A,
2. $\angle AB^\star C^\star \simeq \angle A'B'C'$, and
3. $A^\star B^\star \simeq A'B'$.



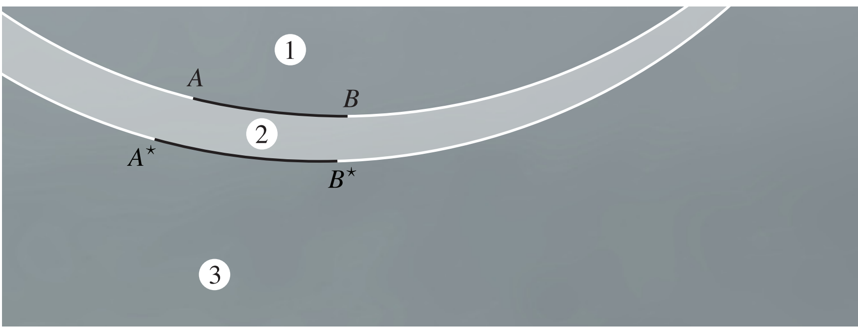
The setup for the proof of AAASS for convex quadrilaterals.

The angle and segment construction axioms guarantee that there is one and only one point that satisfies these conditions. That finishes the copying– by S·A·S·A·S, $\square A^\star B^\star CD$ and $\square A'B'C'D'$ are congruent. There is one important thing to note about this construction. Since

$$\angle A^\star B^\star C \simeq \angle A'B'C' \simeq \angle ABC,$$

the Alternate Interior Angle Theorem guarantees that $\leftarrow A^\star B^\star \rightarrow$ and $\leftarrow AB \rightarrow$ will be parallel.

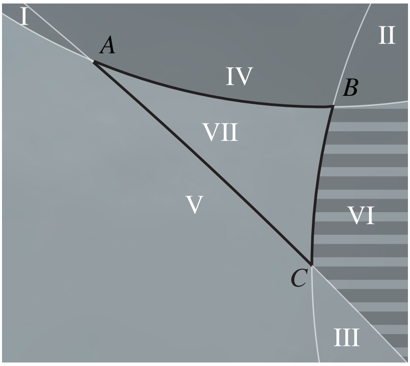
Part two, in which we determine the position of D relative to those lines. The two parallel lines $\leftarrow AB \rightarrow$ and $\leftarrow A^\star B^\star \rightarrow$ carve the plane into three regions as shown in the illustration below. The reason I mention this is

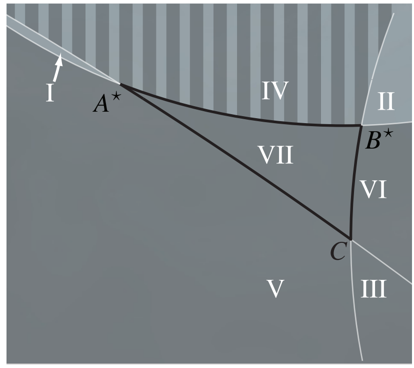


Regions between parallel lines.

that my proof will not work if D is in region 2, the region between the two parallel lines. Now it is pretty easy to show that D will not fall in region 2 if we know the two quadrilaterals are convex. If we don't know that, though, the situation gets a little more delicate, and we will have to look for possible reflex angles in the two quadrilaterals. The key thing to keep in mind is that the angle sum of a simple quadrilateral is at most 360° (a consequence of the Saccheri-Legendre Theorem), and the measure of a reflex angle is more than 180°– therefore, a simple quadrilateral will support at most one reflex angle.

Suppose that D did lie in region 2. Note that, based upon our construction, either $C * B * B^\star$ or $C * B^\star * B$, and so that means that C is *not* in region 2. Therefore, one of the two lines (either $\leftarrow AB \rightarrow$ or $\leftarrow A^\star B^\star \rightarrow$) comes between C and D while the other does not. The two cases are equivalent, so in the interest of keeping the notation reasonable, let's assume for the rest of this proof that $\leftarrow A^\star B^\star \rightarrow$ separates C and D, but that $\leftarrow AB \rightarrow$ does not. What are the implications of this? Let me refer you back to Table 2 which characterizes the possible positions of a fourth vertex of a quadrilateral in relation to the previous three.



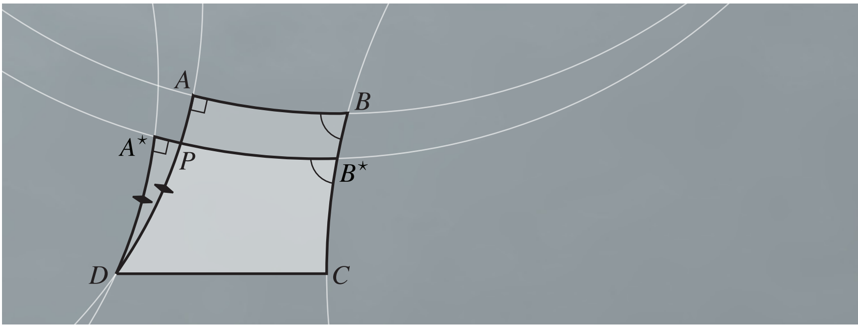


Since C and D are on the same side of $\leftarrow AB \rightarrow$, D has to be in region III, IV, or V with respect to $\triangle ABC$ (note that if D is in region VI, then $\square ABCD$ is not simple). If D is in region III, then $\square ABCD$ has a reflex angle at C. If D is in region V, then $\square ABCD$ is convex and does not have a reflex angle. And if D is in region VII, then $\square ABCD$ has a reflex angle at D.

Since C and D are on opposite sides $\leftarrow A^\star B^\star \rightarrow$, D has to be in region I or II (if D is in region IV, then $\square A^\star B^\star CD$ is not simple. If D is in region I, then $\square A^\star B^\star CD$ has a reflex angle at $A^\star$. If D is in region II, then $\square A^\star B^\star CD$ has a reflex angle at $B^\star$.

A quadrilateral can only have one reflex angle, so in $\square ABCD$ neither $\angle A$ nor $\angle B$ is reflex. In $\square A^\star B^\star CD$ one of $\angle A^\star$ or $\angle B^\star$ is reflex. Remember though that $\angle A^\star \simeq \angle A$ and $\angle B^\star \simeq \angle B$. This is a contradiction– obviously two angles cannot be congruent if one has a measure over 180° while the other has a measure less than that. So now we know that D cannot lie between $\leftarrow AB \rightarrow$ and $\leftarrow A^\star B^\star \rightarrow$ and so *all* the points of $\leftarrow AB \rightarrow$ are on the opposite side of $\leftarrow A^\star B^\star \rightarrow$ from D.

Part Three, in which we measure the distance from D to those lines.
I would like to divide the rest of the proof into two cases. The first case deals with the situation when $\angle A$ and $\angle A^\star$ (which are congruent) are right angles. The second deals with the situation where they are not.

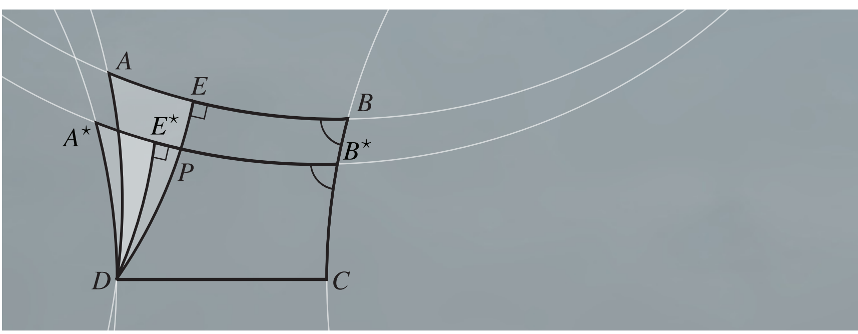


Case 1: the angle at A is a right angle.

Case 1. $(\angle A) = (\angle A^\star) = 90°$.
Since D and A are on opposite sides of $\leftarrow A^\star B^\star \rightarrow$, there is a point P between A and D which is on $\leftarrow A^\star B^\star \rightarrow$. Then

$$|DP| < |DA| = |DA^\star|.$$

But that can't happen, since $A^\star$ is the closest point on $\leftarrow A^\star B^\star \rightarrow$ to D.



Case 2: the angle at A is not a right angle.

Case 2. $(\angle A) = (\angle A^\star) \neq 90°$.
The approach here is quite similar to the one in Case 1. The difference is that we are going to have to make the right angles first. Locate E and $E^\star$, the feet of the perpendiculars from D to $\leftarrow AB \rightarrow$ and $\leftarrow A^\star B^\star \rightarrow$, respectively. Please turn your attention to triangles $\triangle DAE$ and $\triangle DA^\star E^\star$. In them,

$$AD \simeq A^\star D \quad \angle A \simeq \angle A^\star \quad \angle E \simeq \angle E^\star.$$

By A·A·S, they are congruent, and that means that $DE \simeq DE^\star$. But that creates essentially the same problem that we saw in the first case. Since D

and E are on opposite sides of $\leftarrow A^\star B^\star \rightarrow$, there is a point P between D and E which is on $\leftarrow A^\star E^\star \rightarrow$. Then

$$|DP| < |DE| = |DE^\star|.$$

Again, this cannot happen, as $E^\star$ should be the closest point to D on $\leftarrow A^\star E^\star \rightarrow$.

In either case, we have reached a contradiction. The initial assumption, that $\square ABCD$ and $\square A'B'C'D'$ are *not* congruent, must be false.

$\square$

Exercises

1. A convex quadrilateral with two pairs of congruent adjacent sides is called a *kite*. Prove that the diagonals of a kite are perpendicular to one another.

2. Prove the A·S·A·S·A, and A·A·S·A·S quadrilateral congruence theorems.

3. Prove the S·S·S·S·A quadrilateral congruence theorem for *convex* quadrilaterals.

4. Provide Euclidean counterexamples for each of A·S·A·A·S, A·S·A·S·S, A·S·S·A·S, and A·A·A·A·S.

5. Here is another way that you could count words: there are four angles and four sides, a total of eight pieces of information, and you need to choose five of them. That means there are

$$\binom{8}{5} = \frac{8!}{5!(8-5)!} = 56$$

possibilities. That's quite a few more than the $2^5 = 32$ possibilities that I discussed. Resolve this discrepancy and make sure that I haven't missed any congruence theorems.