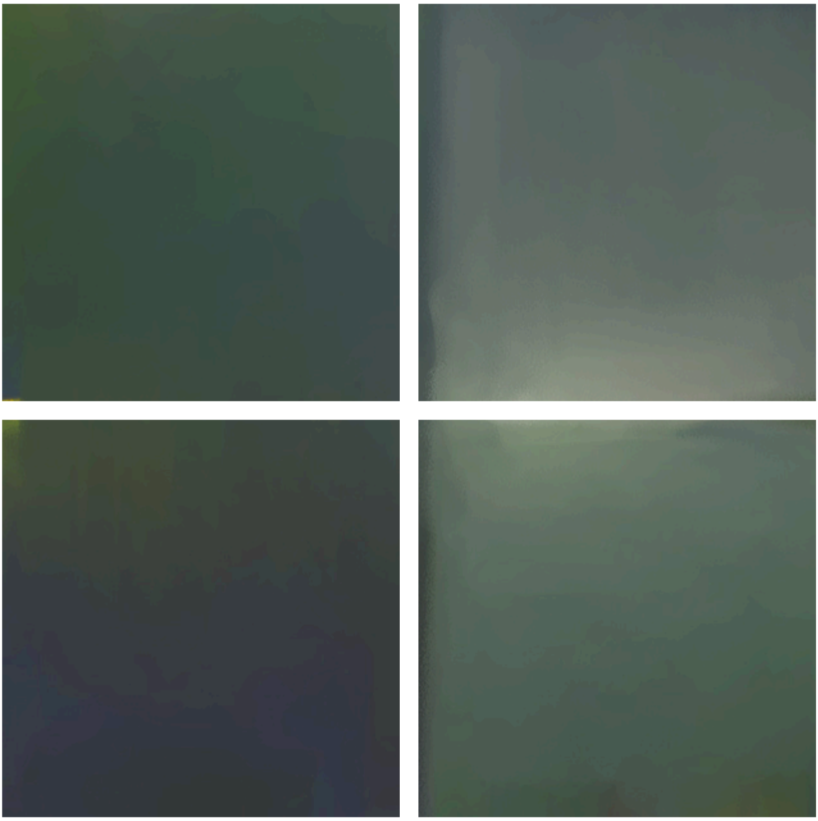
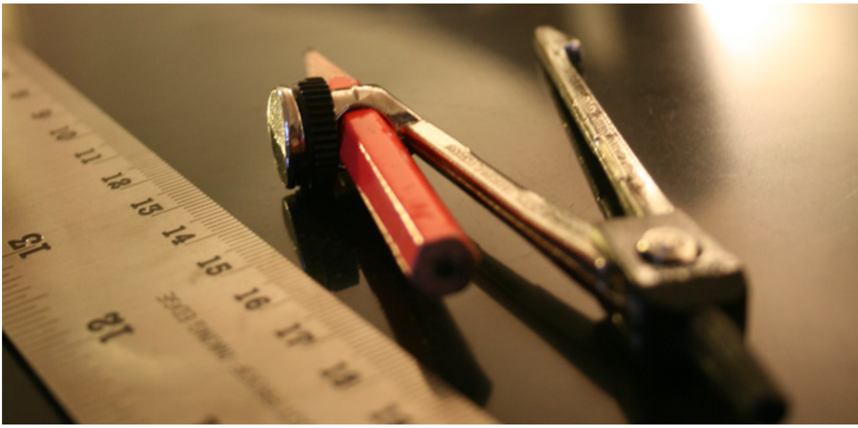


18 THE BLANK CANVAS AWAITS

EUCLIDEAN CONSTRUCTIONS



This lesson is a diversion from our projected path, but I maintain that it is a pleasant and worthwhile diversion. We get a break from the heavy proofs, and we get a much more tactile approach to the subject. I have found that compass and straight edge constructions serve as a wonderful training ground for the rigors of mathematics without the tricky logical pitfalls of formal proof. In my geometry classes, I often don't have time to prove many of the really neat Euclidean results that we will see in the next few lessons, but I have found that I can use compass and straight edge constructions to present the theorems in an sensible way.

Now kindly rewind all the way back to Lesson 1, when I talked briefly about Euclid's postulates. In particular, I want you to look at the first three

P1 To draw a straight line from any point to any point.

P2 To produce a finite straight line continuously in a straight line.

P3 To describe a circle with any center and distance.

Back then, I interpreted these postulates as claims of existence (of lines and circles). Consider instead a more literal reading: they are not claiming the existence of objects, but rather telling us that we can *make* them. This lesson is dedicated to doing just that: constructing geometric objects using two classical tools, a compass and a straight edge. The compass makes circles and arcs, and the straight edge makes segments, rays, and lines. Together they make the kinds of shapes that Euclid promised in his postulates.

The straight edge

The straight edge is a simple tool– it is just something that can draw lines. In all likelihood, your straight edge will be a ruler, and if so, you need to be aware of the key distinction between a ruler and a straight edge. Unlike a ruler, a straight edge has no markings (nor can you add any). Therefore, you cannot measure distance with it. But a straight edge *can* do the following :

– draw a segment between two points;
– draw a ray from a point through another point;
– draw a line through two points;
– extend a segment to either a ray or the line containing it;
– extend a ray to the line containing it.

The compass

Not to be confused with the ever-northward-pointing navigational compass, the compass of geometry is a tool for creating a circle. More precisely, a compass can do the following:

◦ given two distinct points P and Q, draw the circle centered at P which passes through Q;
◦ given points P and Q on a circle with a given center R, draw the arc $\smile PQ$.

You could make a simple compass by tying a pencil to a piece of string, but it would be pretty inaccurate. The metal compasses of my youth (such as the one pictured) are more precise instruments, but alas double as weaponry in the hands of some mischievous rascals. The plastic compasses that are now the norm in many schools are an adequate substitute until they fall apart, usually about halfway through the lesson.

Let me give a warning about something a compass cannot do (at least not "out of the box"). A common temptation is to try to use the compass to transfer distance. That is, to draw a circle of a certain radius, lift up the compass and move it to another location, then place it back down to draw another circle with the same radius. That process effectively transfers a distance (the radius) from one location to another, and so is a convenient way to construct a congruent copy of one segment in another location. It is a simple enough maneuver, but the problem is that according to the classical rules of the game a compass does not have this transfer ability.

The classical compass is “collapsing”, meaning that as soon as it is used to create a circle, it falls apart (in this way, I guess the classical compass does resemble those shoddy plastic ones). We will soon see that the two types of compasses are *not* fundamentally different, and therefore that the non-collapsing feature is actually only a convenience. Once we have shown that, I will have no qualms about using a non-collapsing compass when it will streamline the construction process. Until then, distance transfer using a compass is off-limits.

The digital compass and straight edge

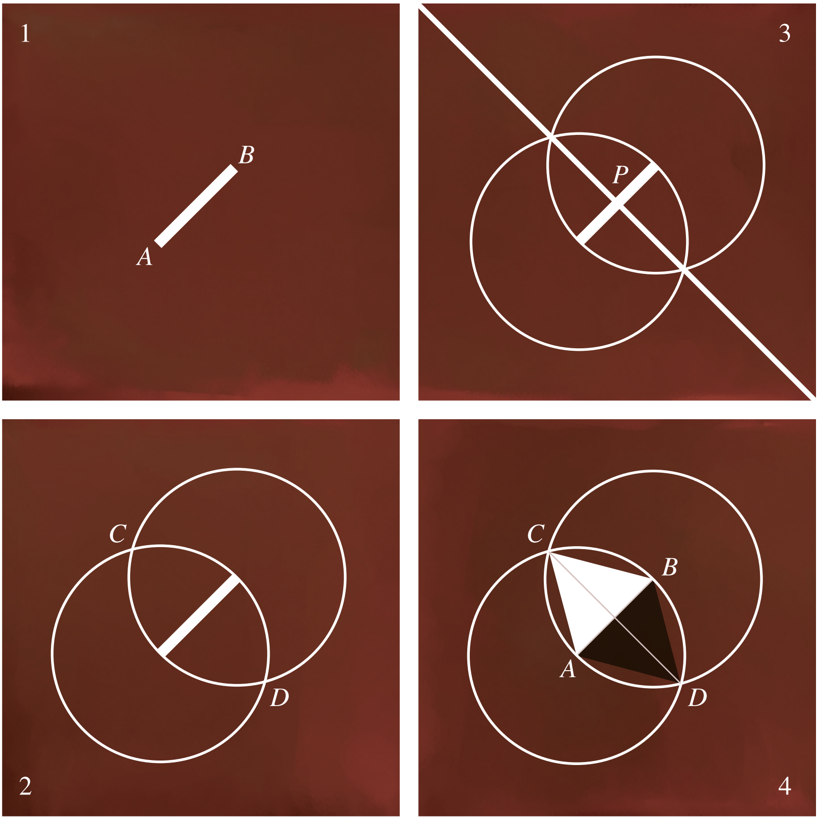
There are several good computer programs that will allow you to build these constructions digitally (though I won’t formally endorse a particular one). There are both advantages and disadvantages to the digital approach. At the risk of sounding like a mystic, I believe that drawing lines and circles on a real piece of paper with a real pencil links you to a long, beautiful tradition in a way that no computer experience can. For more complicated constructions, though, the paper and pencil approach gets really messy. In addition, a construction on paper is static, while computer constructions are dynamic– you can drag points around and watch the rest of the construction adjust accordingly. Often that dynamism really reveals the power of the theorems in a way that no single static image ever could. I would recommend that you try to make a few of the simpler constructions the old-fashioned way, with pencil and paper. And I would recommend that you try a few of the more complicated constructions with the aid of a computer.

A little advice

1. It is easier to draw than to erase.

2. Lines are infinite, but your use for them may not be– try not to draw more of the line than is needed. Similarly, if you only need a small arc of a circle, there is little point in drawing the whole thing.

3. To the extent that you can plan ahead, you can build your construction so that it is neither too big nor too small. The Euclidean plane is infinite, but your piece of paper is not. At the other extreme, your real world compass likely will not function well below a certain radius.



The perpendicular bisector

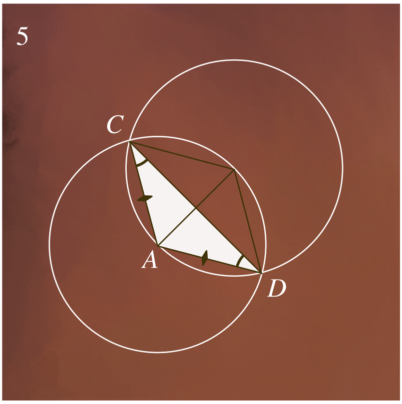
1 Begin with a segment AB.

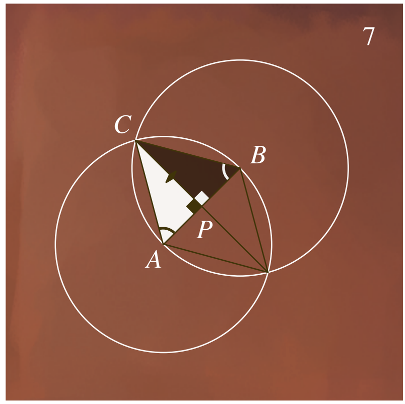
2 With the compass construct two circles: one centered at A which passes through B and one centered at B which passes through A. These circles intersect twice, at C and D, once on each side of AB.

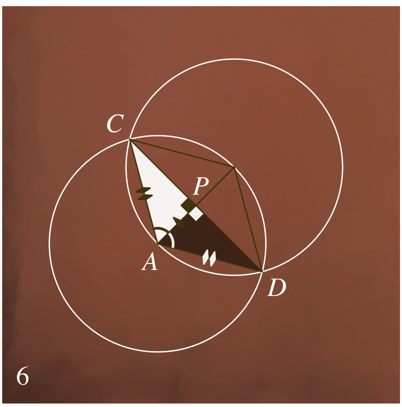
3 Use the straight edge to draw the line $\leftarrow CD \rightarrow$. That line is the perpendicular bisector of AB, and its intersection P with AB is the midpoint of AB.

Perhaps some justification of the last statement is in order. Observe the following.

4 That $\triangle ABC$ and $\triangle ABD$ are equilateral, and since they share a side, are congruent.



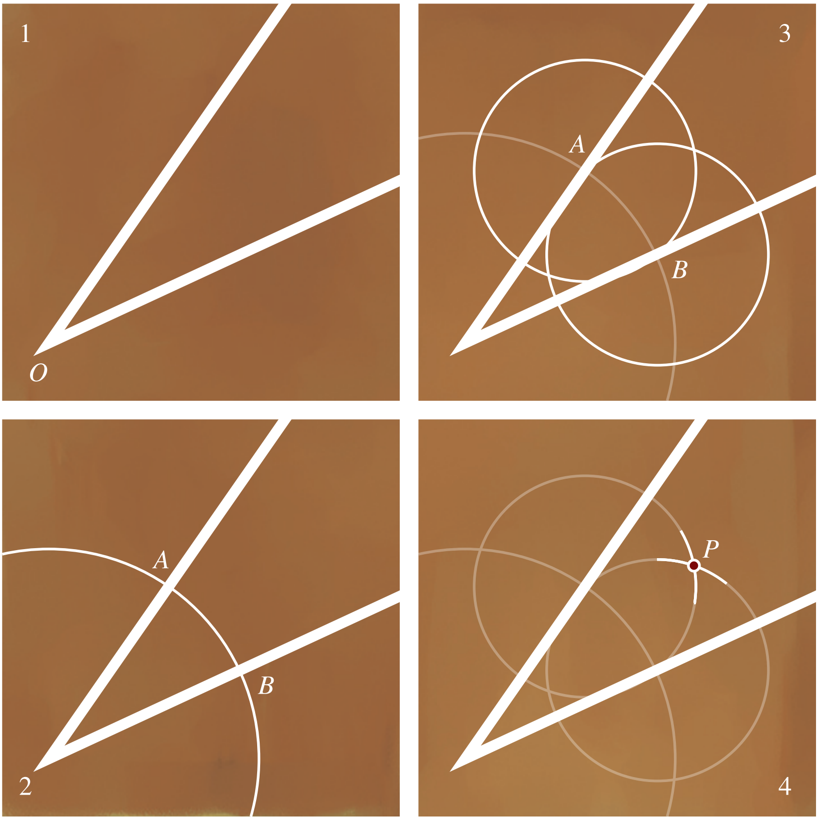




5 That $\triangle ACD$ is isosceles, so the angles opposite its congruent sides, $\angle ACD$ and $\angle ADC$, are congruent.

6 S·A·S: That $\triangle ACP$ and $\triangle ADP$ are congruent. This means $\angle APC$ is congruent to its own supplement, and so is a right angle. That handles the first part of the claim: CD is perpendicular to AB.

7 Continuing, $\angle APC$ and $\angle BPC$ are right angles. By A·A·S, $\triangle APC$ and $\triangle BPC$ are congruent and so $AP \simeq BP$. That means P has to be the midpoint of AB.



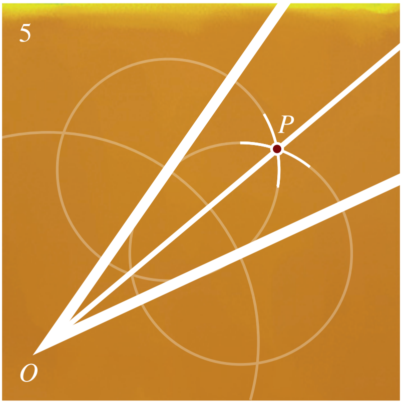
The bisector of an angle

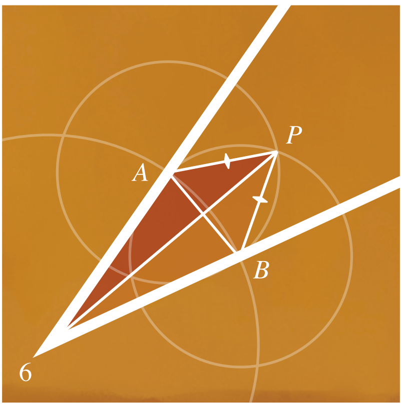
1 Begin with an angle whose vertex is O.

2 Draw a circle centered at O, and mark where it intersects the rays that form the angle as A and B.

3 Draw two circles– one centered at A passing through B, and one centered at B passing through A.

4 Label their intersection as P.



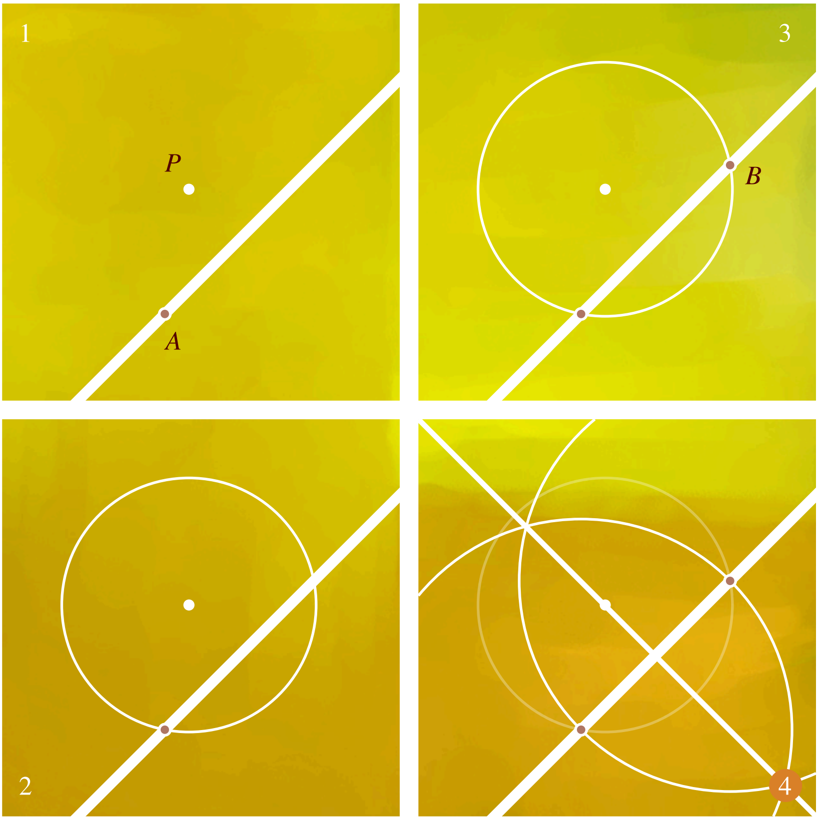


5 Draw the ray $OP \rightarrow$. It is the bisector of $\angle AOB$.

6 The justification is easier this time. You see,

$$AP \simeq AB \simeq BP$$

so by S·S·S, $\triangle OAP \simeq \triangle OBP$. Now match up the congruent interior angles, and $\angle AOP \simeq \angle BOP$.



The perpendicular to a line ℓ through a point P.

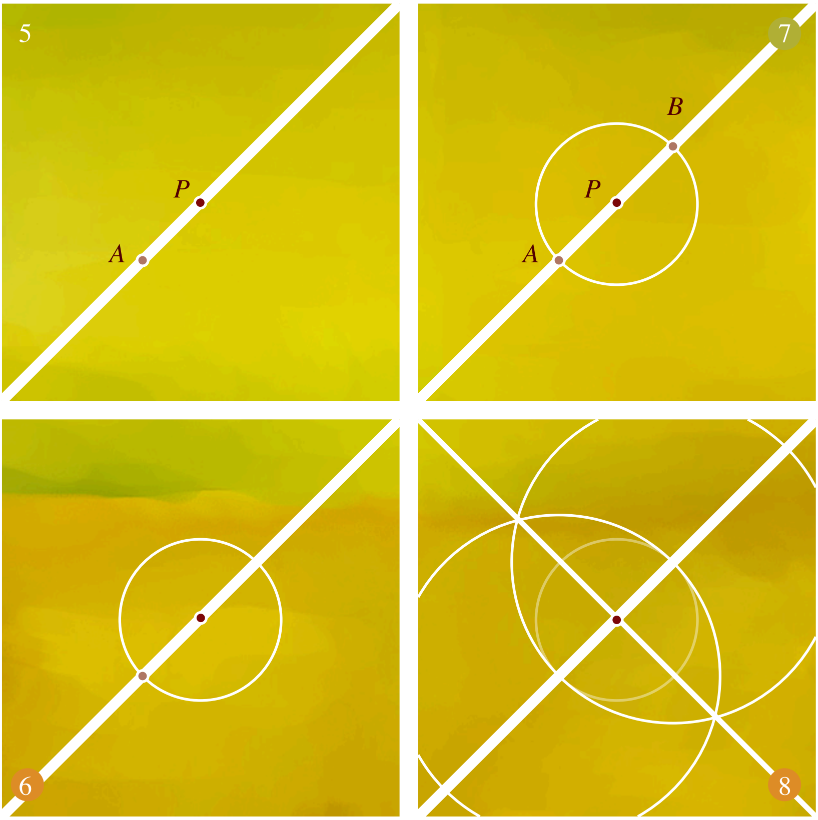
Case 1: if P is not on ℓ

1 Mark a point A on ℓ.

2 Draw the circle centered at P and passing through A.

3 If this circle intersects ℓ only once (at P), then ℓ is tangent to the circle and AP is the perpendicular to ℓ through P (highly unlikely). Otherwise, label the second intersection B.

4 Use the previous construction to find the perpendicular bisector to AB. This is the line we want.



Case 2: if P is on ℓ

5 Mark a point A on ℓ other than P.

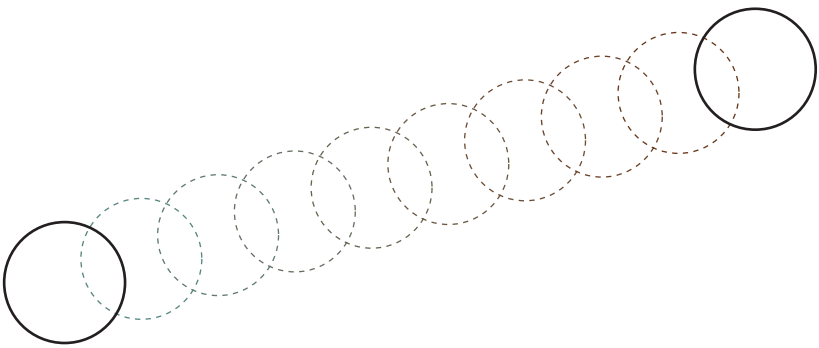
6 Draw the circle centered at P passing through A.

7 Mark the second intersection of this circle with ℓ as B.

8 Use the previous construction to find the perpendicular bisector to AB. This is the line we want.

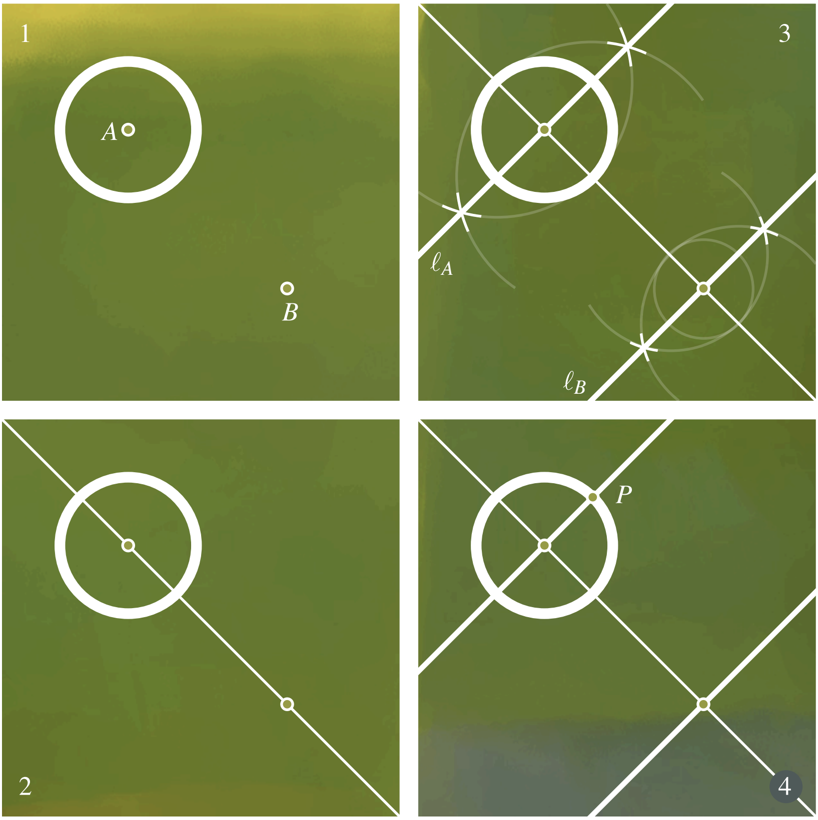
Again, there may be some question about why these constructions work. This time I am going to leave the proof to you.

Once you know how to construct perpendicular lines, constructing parallels is straightforward: starting from any line, construct a perpendicular, and then a perpendicular to that. According to the Alternate Interior Angle Theorem, the result will be parallel to the initial line. Such a construction requires quite a few steps, though, and drawing parallels feels like it should be a fairly simple procedure. As a matter of fact, there is a quicker way, but it requires a non-collapsing compass. So it is now time to look into the issue of collapsing versus non-collapsing compasses.



Collapsing v. non-collapsing

The apparent difference between a collapsing and a non-collapsing compass is that with a non-collapsing compass, we can draw a circle, move the compass to another location, and draw another circle of the same size. In effect, the non-collapsing compass becomes a mechanism for relaying information about size from one location in the plane to another. As I mentioned at the start of this lesson, the official rulebook does not permit a compass to retain and transfer that kind of information. The good news is that, in spite of this added feature, a non-collapsing compass is not any more powerful than a collapsing one. Everything that can be constructed with a non-collapsing compass can also be constructed with a collapsing one. The reason is simple: a collapsing compass can also transfer a circle from one location to another– it just takes a few more steps.

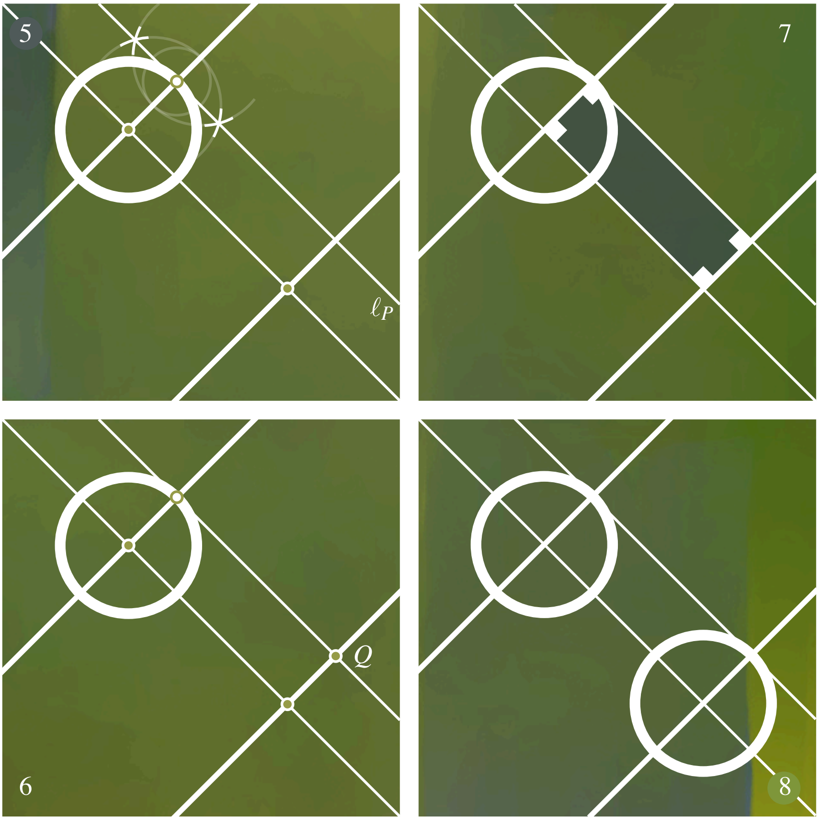


1 Begin with a circle $\mathcal{C}$ with center A. Suppose we wish to draw another circle of the same size, this time centered at a point B.

2 Construct the line $\overleftrightarrow{AB}$.

3 Construct two lines perpendicular to $\overleftrightarrow{AB}$: ℓ_A through A and ℓ_B through B.

4 Now ℓ_A intersects $\mathcal{C}$ twice: identify one point of intersection as P.



5 Construct the line ℓ_P which passes through P and is perpendicular to ℓ_A.

6 This line intersects ℓ_B. Identify the intersection of ℓ_P and ℓ_B as Q.

7 Now A, B, P, and Q are the four corners of a rectangle. The opposite sides AP and BP must be congruent. So finally,

8 Construct the circle with center B which passes through Q. This circle has the same radius as $\mathcal{C}$.

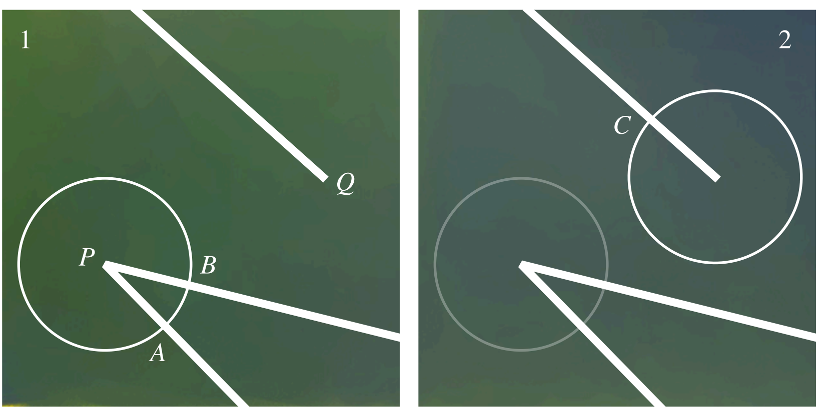
This means that a collapsing compass can do all the same things a non-collapsing compass can. From now on, let's assume that our compass has the non-collapsing capability.

Transferring segments

Given a segment AB and a ray r whose endpoint is C, it is easy to find the point D on r so that $CD \simeq AB$. Just construct the circle centered at A with radius AB, and then (since the compass is non-collapsing) move the compass to construct a circle centered at C with the same radius. The intersection of this circle and r is D.

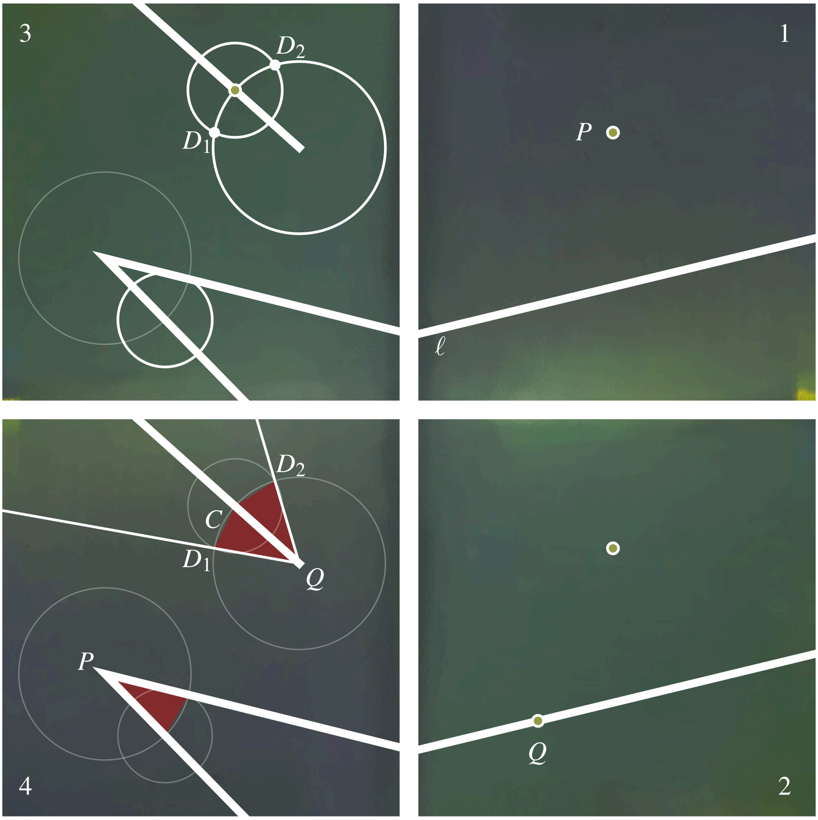
Transferring angles

Transferring a given angle to a new location is a little more complicated. Suppose that we are given an angle with vertex P and a ray r with endpoint Q, and that we want to build congruent copies of $\angle P$ off of r (there are two– one on each side of r).



1 Draw a circle with center P, and label its intersections with the two rays of $\angle P$ as A and B.

2 Using the non-collapsing compass, transfer this circle to one that is centered at Q. Call it $\mathcal{C}$ and label its intersection with r as C.



3 Draw another circle, this time one centered at A which passes through B. Then transfer it to one centered at C. The resulting circle will intersect $\mathcal{C}$ twice, once on each side of r. Label the intersection points as D_1 and D_2.

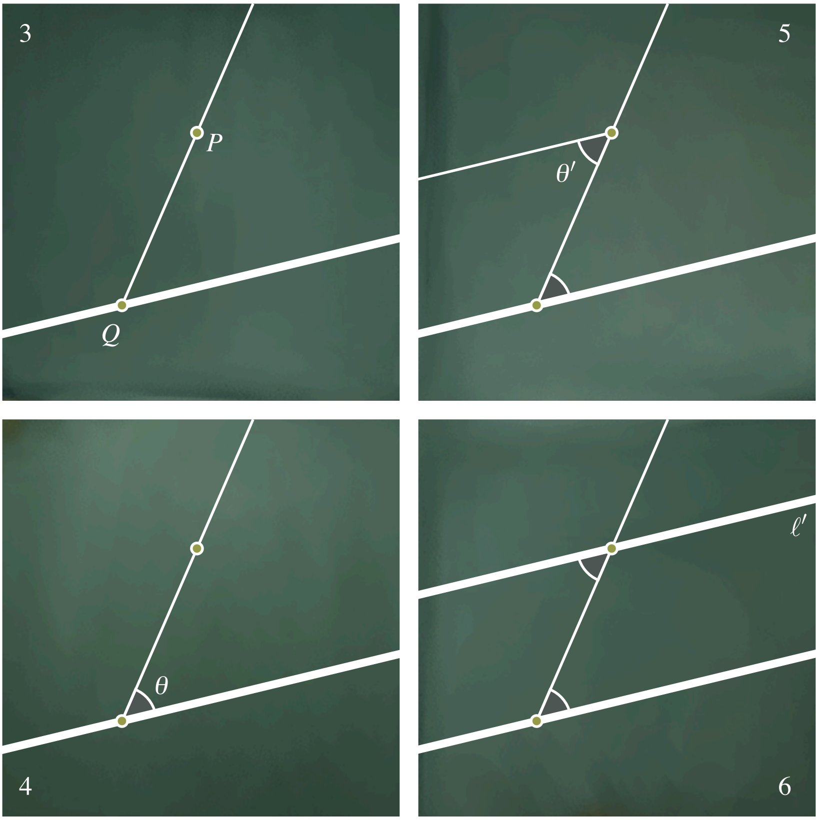
4 By S·S·S, all three of the triangles, $\triangle PAB$, $\triangle PD_1C$, and $\triangle PD_2C$ are congruent. Therefore

$$\angle D_1QC \simeq \angle P \simeq \angle D_2QC.$$

The parallel to a line through a point

1 With a non-collapsing compass and angle transfer, we can now draw parallels the "easy" way. Start with a line ℓ, and a point P which is not on that line.

2 Mark a point Q on ℓ.

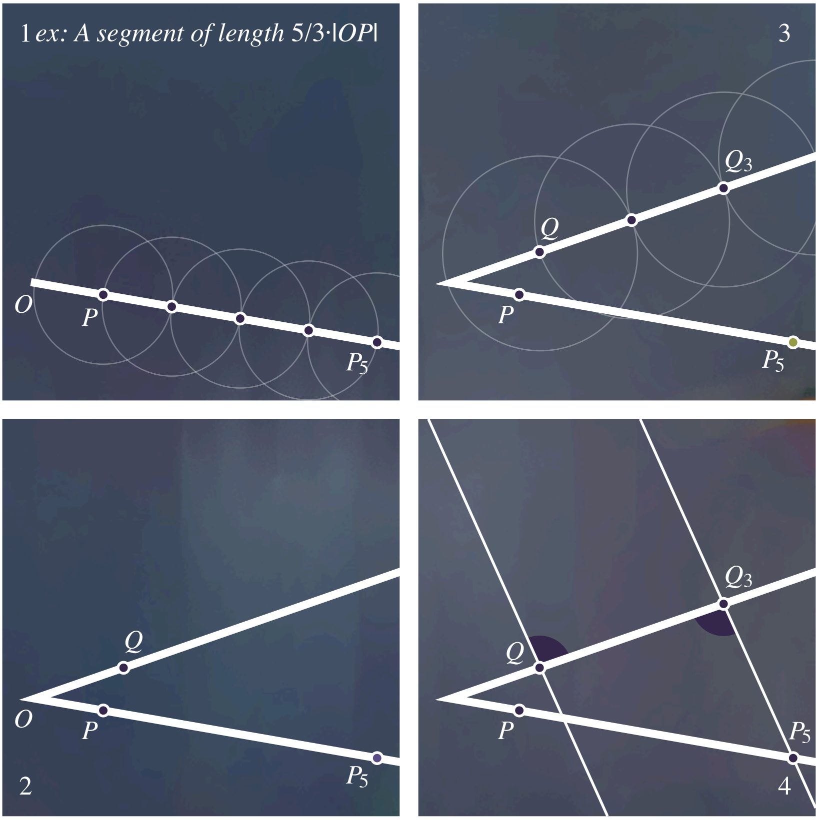


3 Construct the ray $QP\rightarrow$.

4 This ray and ℓ form two angles, one on each side of $QP\rightarrow$. Choose one of these two angles and call it θ.

5 Transfer this angle to another congruent angle θ' which comes off of the ray $PQ\rightarrow$. There are two such angles, one on each side of the ray, but for the purposes of this construction, we want the one on the opposite side of $PQ\rightarrow$ from θ.

6 Now $PQ\rightarrow$ is one of the rays defining θ'. Extend the other ray to the line containing it: call this line ℓ'. By the Alternate Interior Angle Theorem, ℓ' is parallel to ℓ.



A rational multiple of a segment

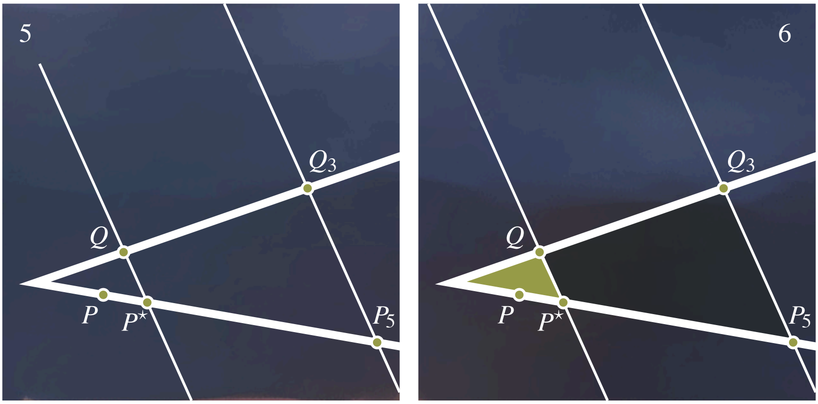
Given a segment OP, we can construct a segment whose length is any rational multiple m/n of $|OP|$.

1 Along $OP\rightarrow$, lay down m congruent copies of OP, end-to-end, to create a segment of length $m|OP|$. Label the endpoint of this segment as P_m.

2 Draw another ray with endpoint O (other than $OP \rightarrow$ or $OP \rightarrow^{op}$), and label a point on it Q.

3 Along $OQ\rightarrow$, lay down n congruent copies of OQ, end-to-end, to create a segment of length $n|OQ|$. Label the endpoint of this segment as Q_n.

4 Draw $\leftarrow P_mQ_n \rightarrow$ and construct the line through Q that is parallel to $\leftarrow P_mQ_n\rightarrow$.



5 It intersects $OP\rightarrow$. Label the intersection as $P^\star$.

6 I claim that $OP^\star$ is the segment we want: that

$$|OP|^\star = m/n \cdot |OP|.$$

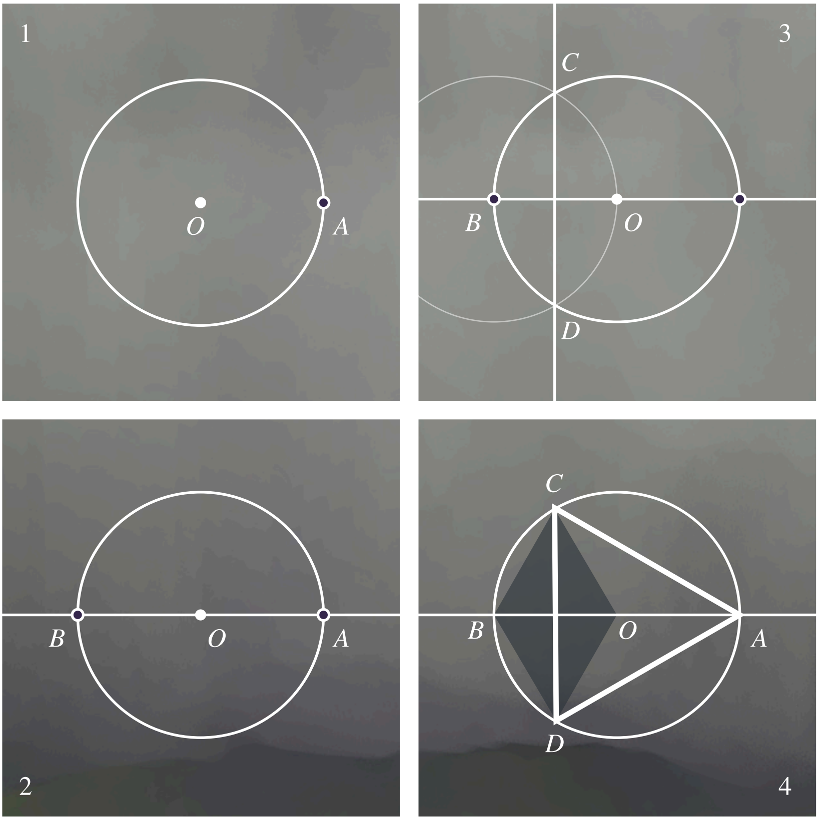
To see why, observe that O, $P^\star$, and P_n are all parallel projections from O, Q, and Q_m, respectively. Therefore,

$$\frac{|OP^\star|}{|OP_m|} = \frac{|OQ|}{|OQ_n|}$$

$$\frac{|OP^\star|}{m \cdot |OP|} = \frac{1}{n}$$

$$|OP^\star| = \frac{m}{n}|OP|.$$

To round out this lesson I would like to look at one of the central questions in the classical theory of constructions: given a circle, is it possible to construct a regular n-gon inscribed in it? This question has now been answered: it turns out that the answer is yes for some values of n, but no for others. In fact, a regular n-gon can be constructed if and only if n is a power of 2, or a product of a power of 2 and distinct Fermat primes (a Fermat prime is a prime of the form $2^{2^n}+1$, and the only known Fermat primes are 3, 5, 17, 257, and 65537). A proof of this result falls outside the scope of this book, but I would like to look at a few of the small values of n where the construction *is* possible. In all cases, the key is to construct a central angle at O which measures $2\pi/n$.



An equilateral triangle that is inscribed in a given circle

In this case, we need to construct a central angle of $2\pi/3$, and this can be done by constructing the supplementary angle of $\pi/3$.

1 Given a circle $\mathcal{C}$ with center O, mark a point A on it.

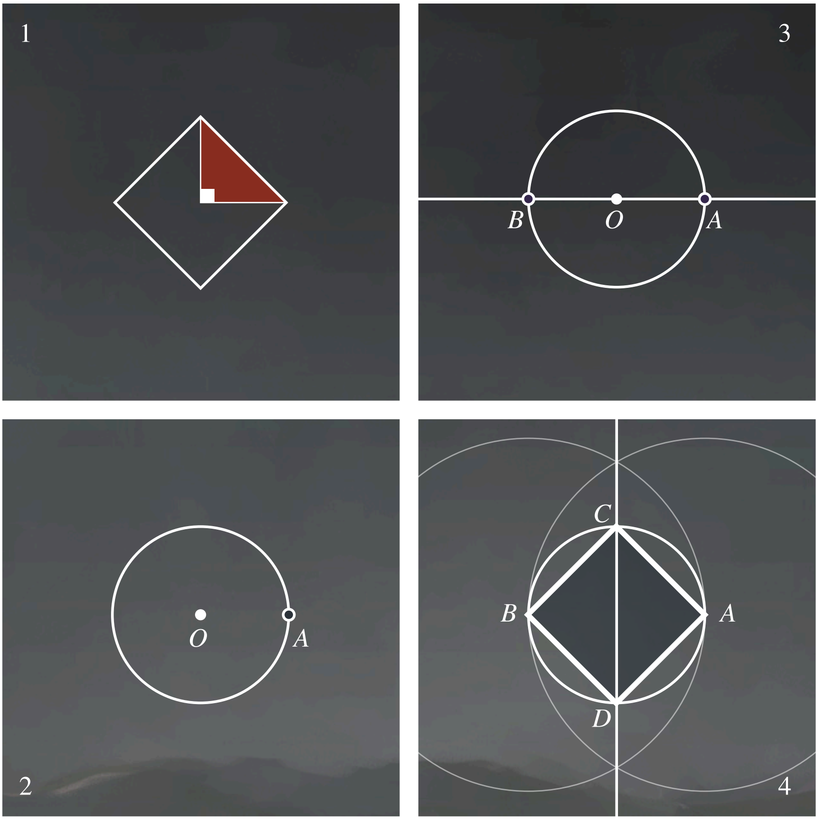
2 Draw the diameter through A, and mark the other endpoint of it as B.

3 Construct the perpendicular bisector to OB. Mark the intersections of that line with $\mathcal{C}$ as C and D.

4 The triangles $\triangle BOC$ and $\triangle BOD$ are equilateral, so

$$(\angle BOC) = (\angle BOD) = \pi/3$$

and so the two supplementary angles $\angle AOC$ and $\angle AOD$ each measure $2\pi/3$. Construct the segments AC and AD to complete the equilateral triangle $\triangle ACD$.



A square inscribed in a given circle

1 This is even easier, since the central angle needs to measure $\pi/2$– a right angle.

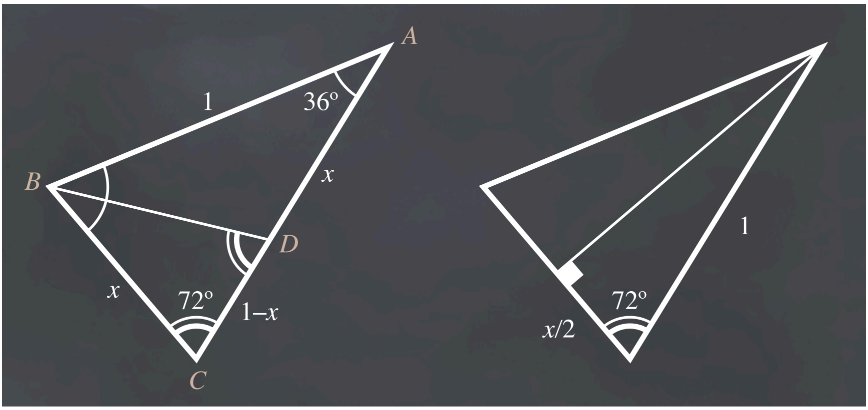
2 Given a circle $\mathcal{C}$ with center O, mark a point A on it.

3 Draw the diameter through A and mark the other endpoint as B.

4 Construct the perpendicular bisector to AB and mark the intersections with $\mathcal{C}$ as C and D. The four points A, B, C, and D are the vertices of the square. Just connect the dots to get the square itself.

A regular pentagon inscribed in a given circle

This one is considerably trickier. The central angle we are going to need is $2\pi/5$ (which is 72°), an angle that you see a lot less frequently than the $2\pi/3$ and the $\pi/2$ of the previous constructions. Before diving into the construction, then, let's take a little time to investigate the geometry of an angle measuring $2\pi/5$. Let me show you a configuration of isosceles triangles that answers a lot of questions.



In this illustration $AB \simeq AC$ and $BC \simeq BD$. Since $\triangle ABC \sim \triangle BCD$, we have a way to solve for x,

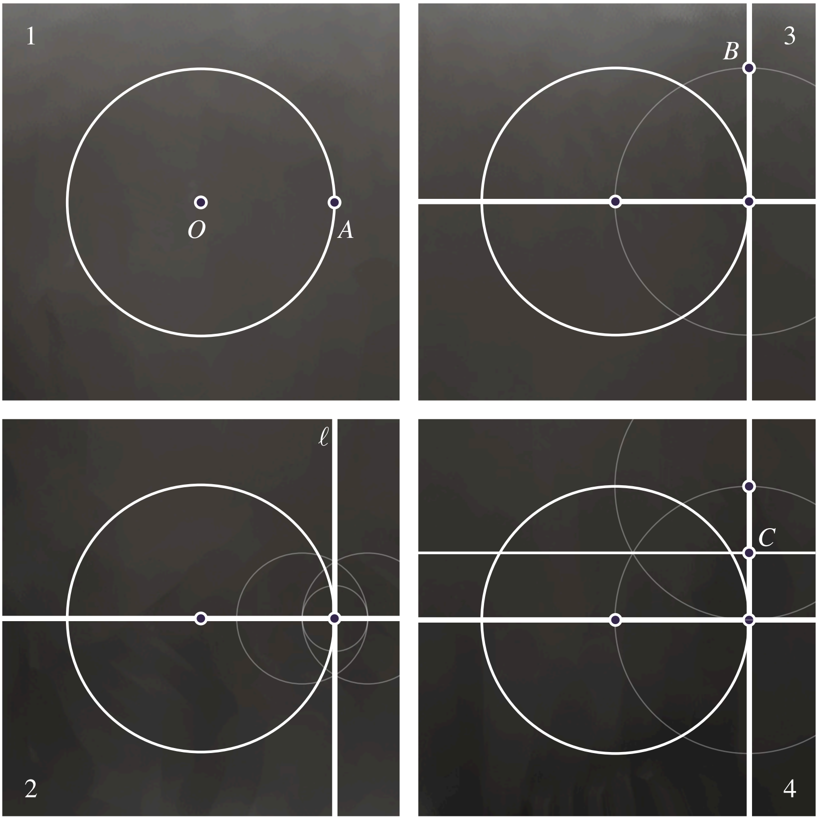
$$\frac{1-x}{x} = \frac{x}{1} \implies 1-x = x^2 \implies x^2 + x - 1 = 0$$

and with the quadratic formula, $x = (-1 \pm \sqrt{5})/2$. Of these solutions, x has to be the positive value since it represents a distance. The line from A to the midpoint of BC divides $\triangle ABC$ into two right triangles, and from them we can read off that

$$\cos(2\pi/5) = \frac{x/2}{1} = \frac{-1+\sqrt{5}}{4}.$$

This cosine value is the key to the construction of the regular pentagon.

[note: I am going with this construction because it seems pretty intuitive, but it is not the most efficient construction. Also, I am going to inscribe this pentagon in a circle of radius one to make the calculations a little easier– the same construction works in a circle of any radius though.]



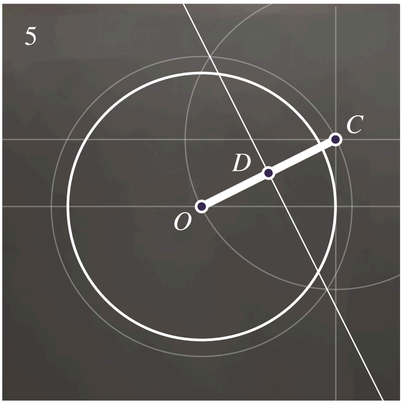
1 Given a circle $\mathcal{C}$ with center O and radius one. Mark a point A on $\mathcal{C}$.

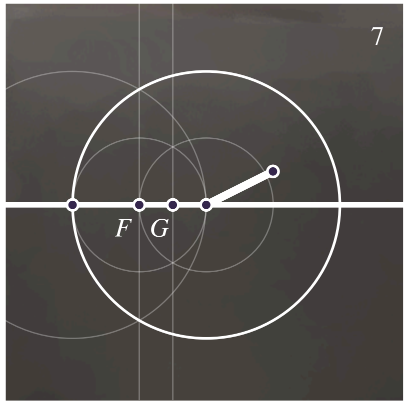
Objective I. Construct a segment of length $\sqrt{5}/4$.

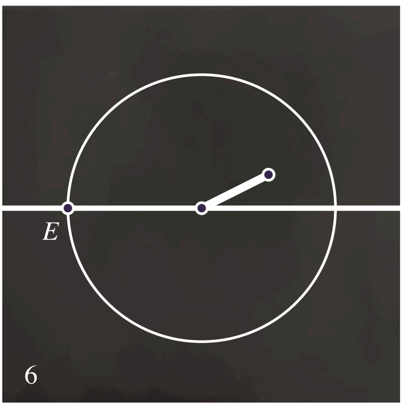
2 Construct the line which passes through A and is perpendicular to $\overleftrightarrow{OA}$. Call this line ℓ.

3 Use the compass to mark a point B on ℓ that is a distance $|OA|$ from A.

4 Construct the midpoint of AB, and call that point C.







5 Draw the segment OC. Note that by the Pythagorean Theorem,

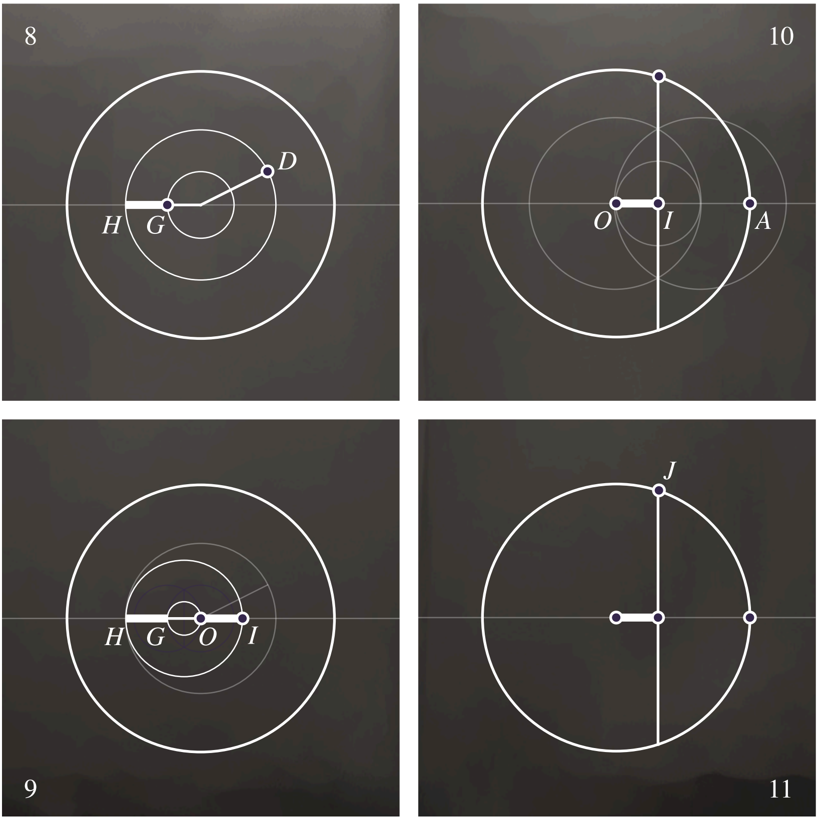
$$\begin{aligned}|OC| &= \sqrt{|OA|^2 + |AC|^2} \\ &= \sqrt{1 + (1/2)^2} \\ &= \sqrt{5}/2.\end{aligned}$$

Locate the midpoint of OC (which is a distance $\sqrt{5}/4$ from O). Call this point D.

Objective II. Construct a segment of length $1/4$.

6 Extend OA until it reaches the other side of $\mathcal{C}$ (the other endpoint of the diameter). Label this point E.

7 Find the midpoint F of OE, and then find the midpoint G of OF. Then $|OE| = 1$, $|OF| = 1/2$ and $|OG| = 1/4$.



Objective III. Construct a segment of length $(-1+\sqrt{5})/4$.

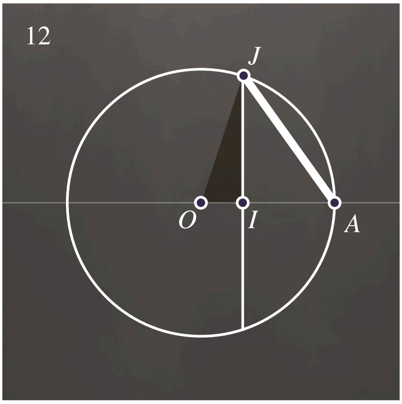
8 Draw the circle centered at O that passes through D. Mark its intersection with OE as H. Then GH is a segment whose length is $(-1+\sqrt{5})/4$.

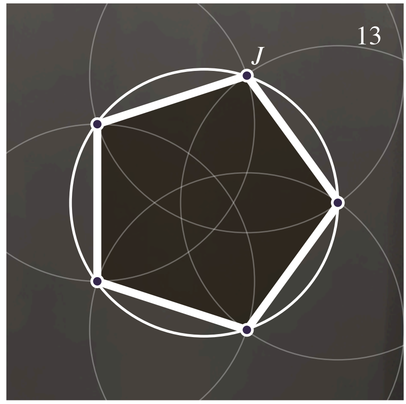
9 Use segment transfer to place a congruent copy of GH along the ray $OA\rightarrow$, with one endpoint at O. Label the other endpoint I.

Objective IV. Mark a vertex of the pentagon.

10 We will use A as one vertex of the pentagon. For the next, construct the line perpendicular to OA which passes through I.

11 Mark one of the intersections of this perpendicular with $\mathcal{C}$ as J.





12 Now look at $\angle O$ in the right triangle $\triangle OIJ$

$$\cos(\angle O) = \frac{|OI|}{|OJ|} = \frac{(-1+\sqrt{5})/4}{1}.$$

According to our previous calculation, that means $(\angle OIJ) = 2\pi/5$.

Objective V. The pentagon itself.

13 Segment AJ is one of the sides of the pentagon. Now just transfer congruent copies of that segment around the circle to get the other four sides of the pentagon.

Exercises

1. Given a segment AB, construct a segment of length $(7/3)|AB|$.
2. In a given circle, construct a regular (i) octagon, (ii) dodecagon, (iii) decagon.
3. Given a circle $\mathcal{C}$ and a point A outside the circle, construct the lines through A that are tangent to $\mathcal{C}$.
4. Foreshadowing. (i) Given a triangle, construct the perpendicular bisectors to the three sides. (ii) Given a triangle, construct the three angle bisectors.

   We haven't discussed area yet, but if you are willing to do some things out of order, here are a few area-based constructions.

5. Given a square whose area is A, construct a square whose area is $2A$.
6. Given a rectangle, construct a square with the same area.
7. Given a triangle, construct a rectangle with the same area.

References

Famously, it is impossible to trisect an angle with compass and straight edge. The proof of this impossibility requires a little Galois Theory, but for the reader who has seen abstract algebra, is quite accessible. Proofs are often given in abstract algebra books– I like Durbin's approach in his *Modern Algebra* book [1](probably because it was the first one I saw).

[1] John R. Durbin. *Modern Algebra: An Introduction*. John Wiley and Sons, Inc., New York, 3rd edition, 1992.