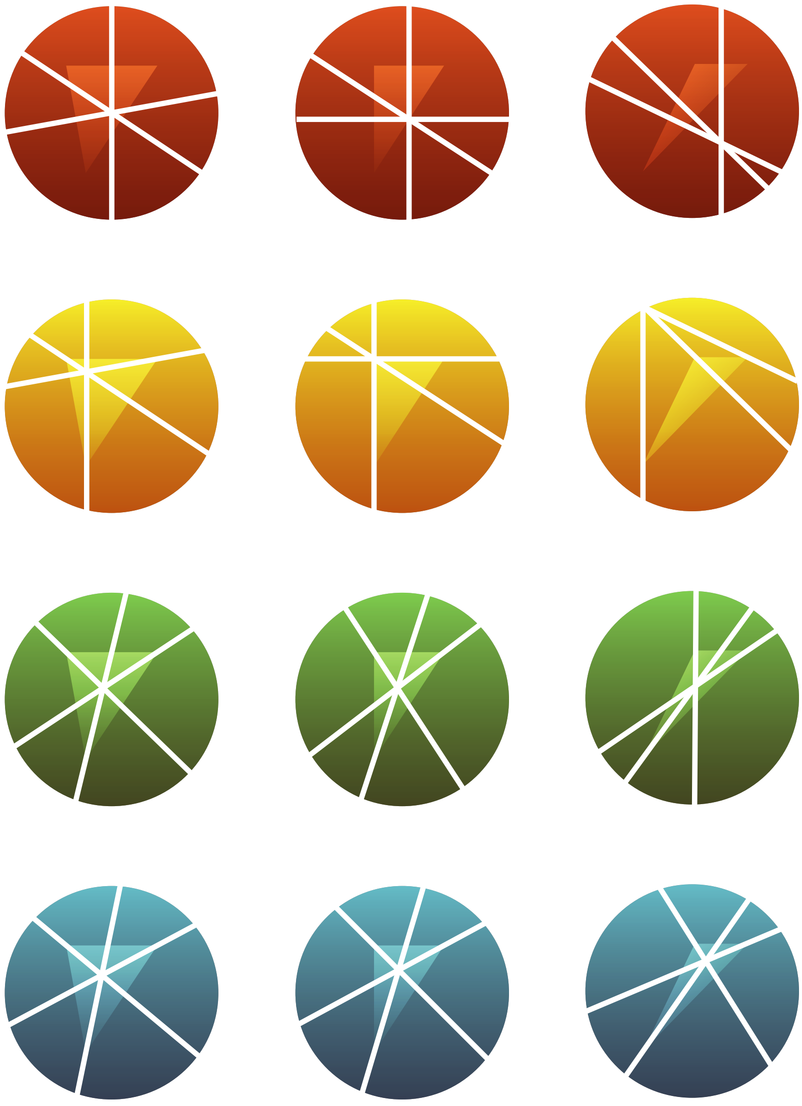
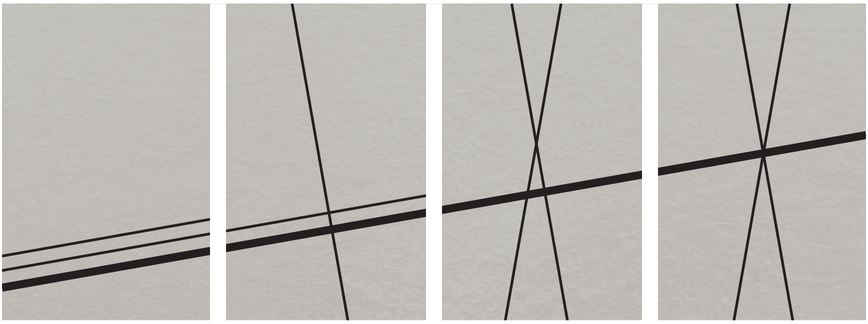


19 CONCURRENCE I



Parallelism *Intersection* *Concurrence*

Start with three (or more) points. There is a small chance that those points all lie on the same line– that they are colinear. In all likelihood, though, they are not. And so, should we find a configuration of points that are consistently colinear, well, that could be a sign of something interesting. Likewise, with three (or more) lines, the greatest likelihood is that each pair of lines interect, but that none of the intersections coincide. It is unusual for two lines to be parallel, and it is unusual for three or more lines to intersect at the same point.

DEF: CONCURRENCE
When three (or more) lines all intersect at the same point, the lines are said to be *concurrent*. The intersection point is called the *point of concurrence*.

In this lesson we are going to look at a few (four) concurrences of lines associated with a triangle. Geometers have catalogued thousands of these concurrences, so this is just the tip of a very substantial iceberg. [1]

The circumcenter

In the last lesson, I gave the construction of the perpendicular bisector of a segment, but I am not sure that I ever properly defined it (oops). Let me fix that now.

DEF: PERPENDICULAR BISECTOR
The *perpendicular bisector* of a segment *AB* is the line which is perpendicular to *AB* and passes through its midpoint.

Our first concurrence deals with the perpendicular bisectors of the three sides of a triangle, but in order to properly understand that concurrence, we need another characterization of the points of the perpendicular bisector.

LEMMA

A point X is on the perpendicular bisector to AB if and only if

$$|AX| = |BX|.$$

Proof. There's not much to this proof. It is really just a simple application of some triangle congruence theorems. First, suppose that X is a point on the perpendicular bisector to AB and let M be the midpoint of AB. Then

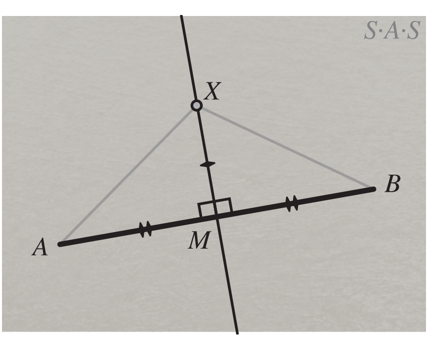
$$S: \ AM \simeq BM$$
$$A: \ \angle AMX \simeq \angle BMX$$
$$S: \ MX = MX,$$

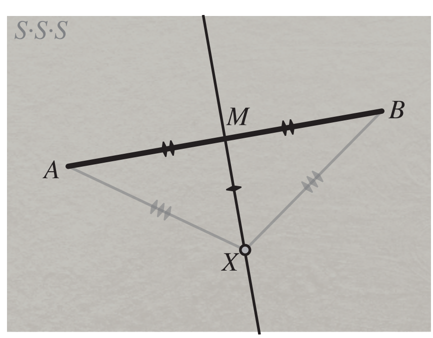
and so $\triangle AMX$ and $\triangle BMX$ are congruent. This means that $|AX| = |BX|$.

Conversely, suppose that $|AX| = |BX|$, and again let M be the midpoint of AB. Then

$$S: \ AM \simeq BM$$
$$S: \ MX = MX$$
$$S: \ AX \simeq BX.$$

and so $\triangle AMX$ and $\triangle BMX$ are congruent. In particular, this means that $\angle AMX \simeq \angle BMX$. Those two angles are supplements, though, and so they must be right angles. Hence X is on the line through M that forms a right angle with AB– it is on the perpendicular bisector. □

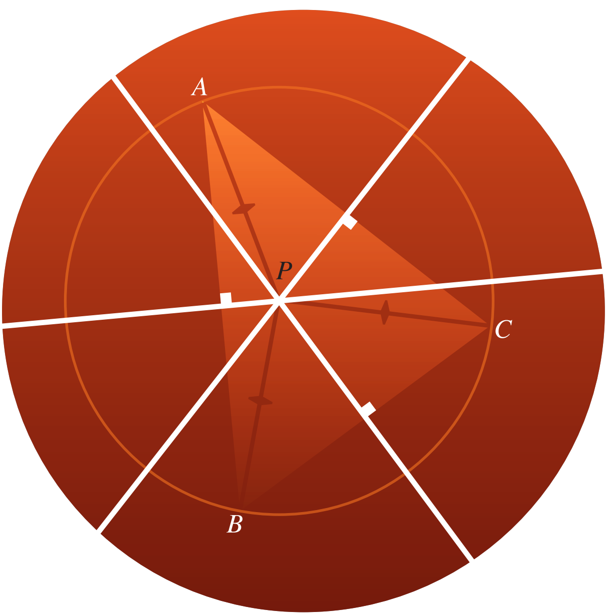




Now we are ready for the first concurrence.

THE CIRCUMCENTER
The perpendicular bisectors to the three sides of a triangle $\triangle ABC$ intersect at a single point. This point of concurrence is called the *circumcenter* of the triangle.

Proof. The first thing to notice is that no two sides of the triangle can be parallel. Therefore, none of the perpendicular bisectors can be parallel–they all intersect each other. Let P be the intersection point of the perpendicular bisectors to AB and BC. Since P is on the perpendicular bisector to AB, $|PA| = |PB|$. Since P is on the perpendicular bisector to BC, $|PB| = |PC|$. Therefore, $|PA| = |PC|$, and so P is on the perpendicular bisector to AC. □

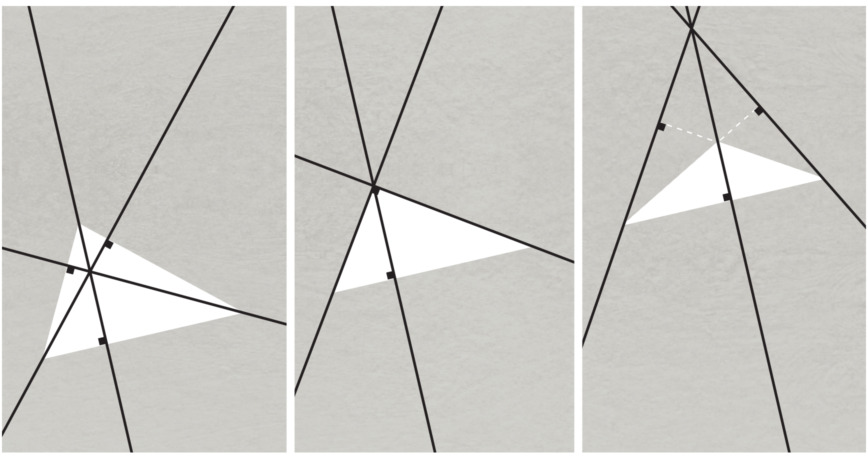


An important side note: P is equidistant from A, B and C. That means that there is a circle centered at P which passes through A, B, and C. This circle is called the *circumcircle* of $\triangle ABC$. In fact, it is the only circle which passes through all three of A, B, and C (which sounds like a good exercise).

The orthocenter

Most people will be familiar with the altitudes of a triangle from area calculations in elementary geometry. Properly defined,

DEF: ALTITUDE
An *altitude* of a triangle is a line which passes through a vertex and is perpendicular to the opposite side.



Altitudes for an acute, right, and obtuse triangle.

You should notice that an altitude of a triangle does not have to pass through the interior of the triangle at all. If the triangle is acute then all three altitudes will cross the triangle interior, but if the triangle is right, two of the altitudes will lie along the legs, and if the triangle is obtuse, two of the altitudes will only touch the triangle at their respective vertices. In any case, though, the altitude from the largest angle *will* cross through the interior of the triangle.

THE ORTHOCENTER
The three altitudes of a triangle $\triangle ABC$ intersect at a single point. This point of concurrence is called the *orthocenter* of the triangle.

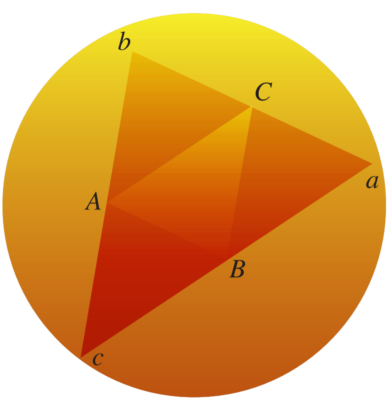
Proof. The key to this proof is that the altitudes of $\triangle ABC$ also serve as the perpendicular bisectors of another (larger) triangle. That takes us back to what we have just shown– that the perpendicular bisectors of a triangle are concurrent. First, we have to build that larger triangle. Draw three lines

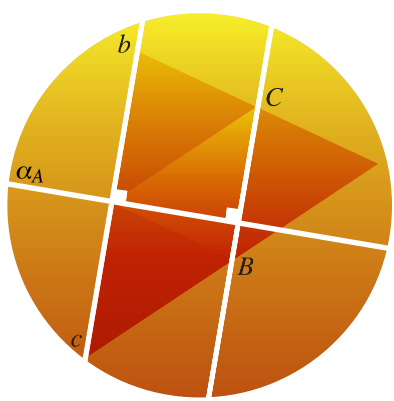
ℓ_1 which passes through A and is parallel to BC,
ℓ_2 which passes through B and is parallel to AC,
ℓ_3 which passes through C and is parallel to AB.

Each pair of those lines intersect (they cannot be parallel since the sides of $\triangle ABC$ are not parallel), for a total of three intersections

$$\ell_1 \cap \ell_2 = c \quad \ell_2 \cap \ell_3 = a \quad \ell_3 \cap \ell_1 = b.$$

The triangle $\triangle abc$ is the "larger triangle". Now we need to show that an altitude of $\triangle ABC$ is a perpendicular bisector of $\triangle abc$. The argument is the same for each altitude (other than letter shuffling), so let's just focus on the altitude through A: call it α_A. I claim that α_A is the perpendicular bisector to bc. There are, of course, two conditions to show: (1) that $\alpha_A \perp bc$ and (2) that their intersection, A, is the midpoint of bc.





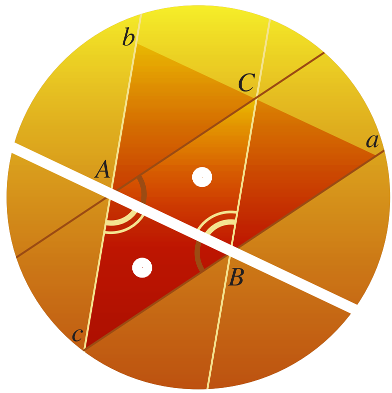
(1) The first is easy thanks to the simple interplay between parallel and perpendicular lines in Euclidean geometry.

$$bc \parallel BC \quad \& \quad BC \perp \alpha_A \quad \Longrightarrow \quad bc \perp \alpha_A.$$

(2) To get at the second, we are going to have to leverage some of the congruent triangles that we have created.

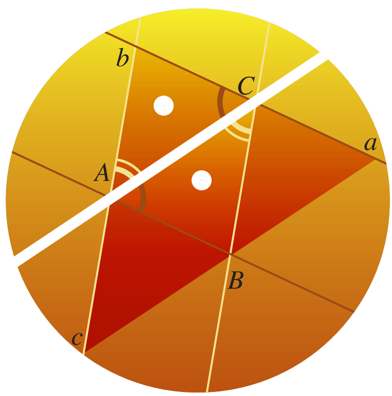
$$\begin{aligned} A:&\quad AC \parallel ac \implies \angle cBA \simeq \angle BAC\\ S:&\quad AB = AB\\ A:&\quad BC \parallel bc \implies \angle cAB \simeq \angle ABC \end{aligned}$$

$$\therefore\quad \triangle ABc \simeq \triangle BAC.$$

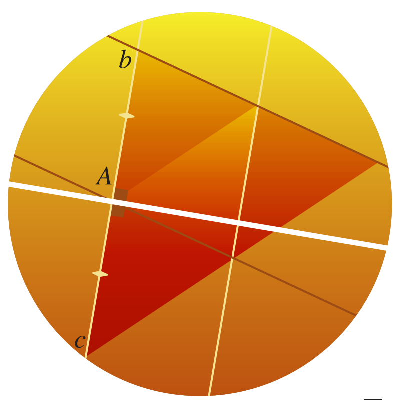


$$\begin{aligned} A:&\quad AB \parallel ab \implies \angle BAC \simeq \angle bCA\\ S:&\quad AC = AC\\ A:&\quad BC \parallel bc \implies \angle BCA \simeq \angle bAC \end{aligned}$$

$$\therefore\quad \triangle ABC \simeq \triangle CbA.$$



Therefore $Ac \simeq BC \simeq Ab$, placing A at the midpoint of bc and making α_A the perpendicular bisector to bc. Likewise, the altitude through B is the perpendicular bisector to ac and the altitude through C is the perpendicular bisector to ab. As the three perpendicular bisectors of $\triangle abc$, these lines must intersect at a single point.



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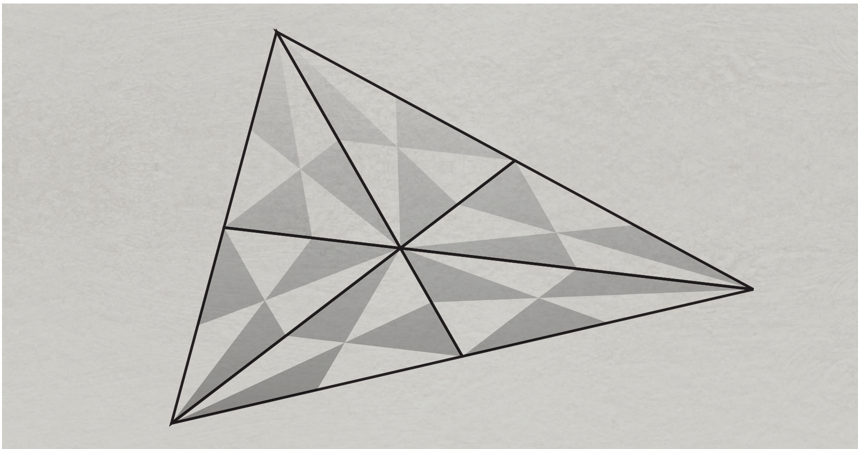


The three medians of a triangle

The centroid

MEDIAN

A *median* of a triangle is a line segment from a vertex to the midpoint of the opposite side.

THE CENTROID

The three medians of a triangle intersect at a single point. This point of concurrence is called the *centroid* of the triangle.

Proof. On $\triangle ABC$, label the midpoints of the three edges,

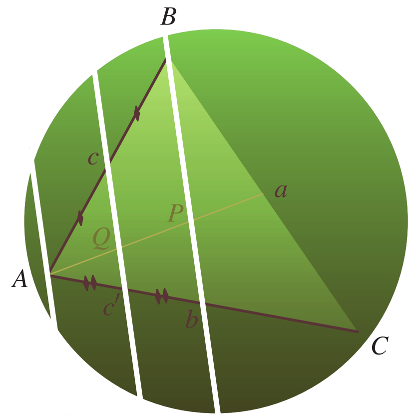
a, the midpoint of BC,
b, the midpoint of AC,
c, the midpoint of AB,

so that Aa, Bb, and Cc are the medians. The key to this proof is that we can pin down the location of the intersection of any two medians– it will always be found two-thirds of the way down the median from the vertex. To understand why this is, we are going to have to look at a sequence of three parallel projections.

1. Label the intersection of Aa and Bb as P. Extend a line from c which is parallel to Bb. Label its intersection with Aa as Q, and its intersection with AC as c'. The first parallel projection, from AB to AC, associates the points

$$A \mapsto A \quad B \mapsto b \quad c \mapsto c'.$$

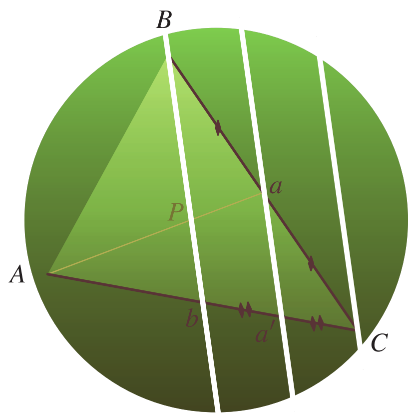
Since $Ac \simeq cB$, this means $Ac' \simeq c'b$.



2. Extend a line from a which is parallel to Bb. Label its intersection with AC as a'. The second parallel projection, from BC to AC, associates the points

$$C \mapsto C \quad B \mapsto b \quad a \mapsto a'.$$

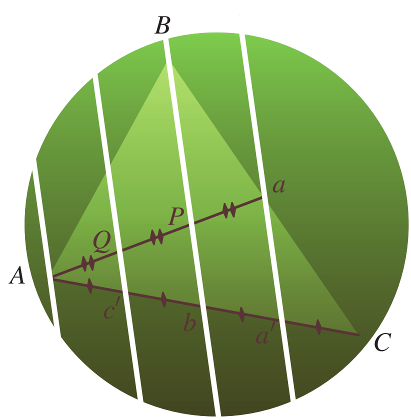
Since $Ca \simeq aB$, this means $Ca' \simeq a'b$.



3. Now b divides AC into two congruent segments, and a' and c' evenly subdivide them. In all, a', b, and c' split AC into four congruent segments. The third parallel projection is from AC back onto Aa:

$$A \mapsto A \quad c' \mapsto Q \quad b \mapsto P \quad a' \mapsto a.$$

Since $Ac' \simeq c'b \simeq ba'$, this means $AQ \simeq QP \simeq Pa$.



Therefore P, the intersection of Bb and Aa, will be found on Aa exactly two-thirds of the way down the median Aa from the vertex A. Now the letters in this argument are entirely arbitrary– with the right permutation of letters, we could show that any pair of medians will intersect at that two-thirds mark. Therefore, Cc will also intersect Aa at P, and so the three medians concur. □

Students who have taken calculus may already be familiar with the centroid (well, probably not my students, since I desperately avoid that section of the book, but students who have more conscientious and responsible teachers). In calculus, the centroid of a planar shape D can be thought of as its balancing point, and its coordinates can be calculated as

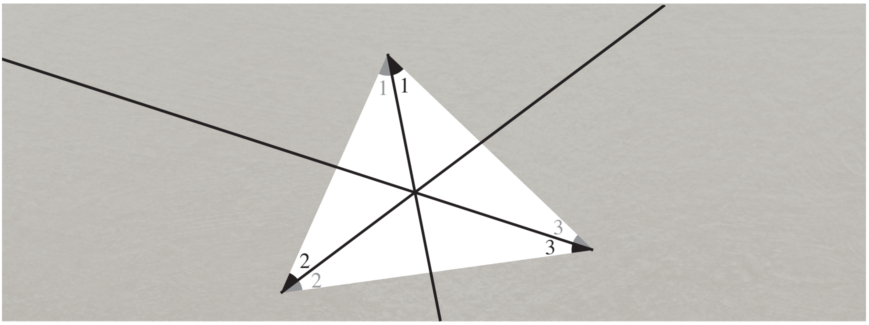
$$\frac{1}{\iint_D 1\,dxdy}\left(\iint_D x\,dxdy,\ \iint_D y\,dxdy\right).$$

It is worth noting (and an exercise for students who have done calculus) that in the case of triangles, the calculus and geometric definitions do coincide.

The incenter

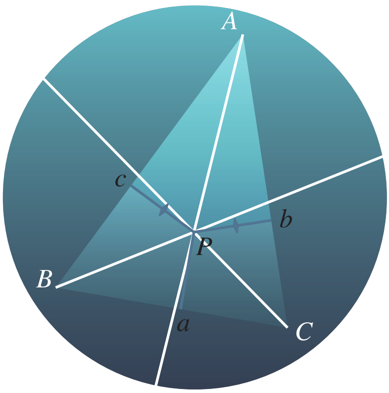
This lesson began with bisectors of the sides of a triangle. It seems fitting to end it with the bisectors of the interior angles of a triangle.

THE INCENTER
The bisectors of the three interior angles of a triangle intersect at a single point. This point of concurrence is called the *incenter* of the triangle.



Angle bisectors

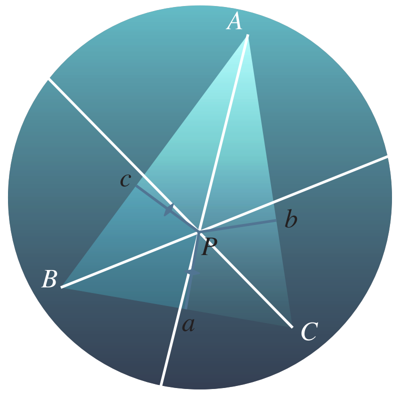
Proof. Take two of the angle bisectors, say the bisectors of $\angle A$ and $\angle B$, and label their intersection as P. We need to show that $CP\rightarrow$ bisects $\angle C$. The key to this proof is that P is actually equidistant from the three sides of $\triangle ABC$. From P, drop perpendiculars to each of the three sides of $\triangle ABC$. Label the feet of those perpendiculars: a on BC, b on AC, and c on AB.



Then

$$\begin{aligned} A:&\quad \angle PbA \simeq \angle PcA\\ A:&\quad \angle bAP \simeq \angle cAP\\ S:&\quad AP = AP \end{aligned}$$

so $\triangle AcP$ is congruent to $\triangle AbP$ and in particular $bP \simeq cP$.

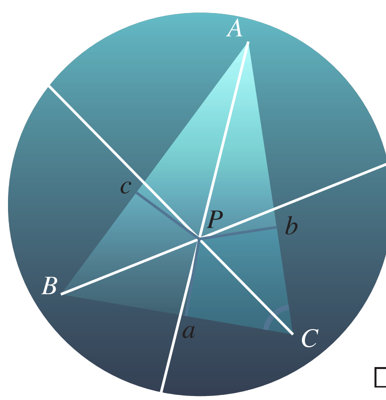


Again,

$$\begin{aligned} A:&\quad \angle PaB \simeq \angle PcB\\ A:&\quad \angle aBP \simeq \angle cBP\\ S:&\quad BP = BP \end{aligned}$$

and so $\triangle BaP$ is congruent to $\triangle BcP$ and in particular $cP \simeq aP$.

Now notice that the two right triangles $\triangle PaC$ and $\triangle PbC$ have congruent legs aP and bP and share the same hypotenuse PC. According to the H·L congruence theorem for right triangles, they have to be congruent. Thus, $\angle aCP \simeq \angle bCP$, and so $CP\rightarrow$ is the bisector of $\angle C$.



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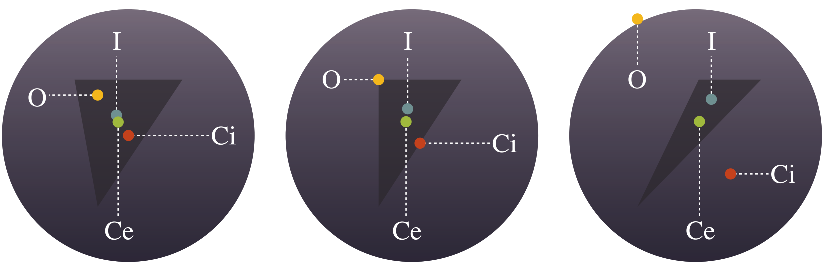
Notice that P is the same distance from each of the three feet a, b, and c. That means that there is a circle centered at P which is tangent to each of the three sides of the triangle. This is called the *inscribed circle*, or *incircle* of the triangle. It is discussed further in the exercises.

References

[1] Clark Kimberling. Encyclopedia of triangle centers - etc. distributed on World Wide Web. http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Triangle Centers



Ci	circumcenter	concurrence of perpendicular bisectors
O	orthocenter	concurrence of altitudes
Ce	centroid	concurrence of medians
I	incenter	concurrence of angle bisectors

Exercises

1. Using only compass and straight edge, construct the circumcenter, orthocenter, centroid, and incenter of a given triangle.

2. Using only compass and straight edge, construct the circumcircle and incircle of a given triangle.

3. Let A, B, and C be three non-colinear points. Show that the circumcircle to $\triangle ABC$ is the only circle passing through all three points A, B, and C.

4. Let A, B and C be three non-colinear points. Show that the incircle is the unique circle which is contained in $\triangle ABC$ and tangent to each of the three sides.

5. Show that the calculus definition and the geometry definition of the centroid of a triangle are the same.

6. Under what circumstances does the circumcenter of a triangle lie outside the triangle? What about the orthocenter?

7. Under what circumstances do the orthocenter and circumcenter coincide? What about the orthocenter and centroid? What about the circumcenter and centroid?

8. For any triangle $\triangle ABC$, there is an associated triangle called the orthic triangle whose three vertices are the feet of the altitudes of $\triangle ABC$. Prove that the orthocenter of $\triangle ABC$ is the incenter of its orthic triangle. [Hint: look for cyclic quadrilaterals and recall that the opposite angles of a cyclic quadrilateral are supplementary.]

9. Suppose that $\triangle ABC$ and $\triangle abc$ are similar triangles, with a scaling constant k, so that $|AB|/|ab| = k$. Let P be a center of $\triangle ABC$ (circumcenter, orthocenter, centroid, or incenter) and let p be the corresponding center of $\triangle abc$. (1) Show that $|AP|/|ap| = k$. (2) Let D denote the distance from P to AB and let d denote the distance from p to ab. Show that $D/d = k$.