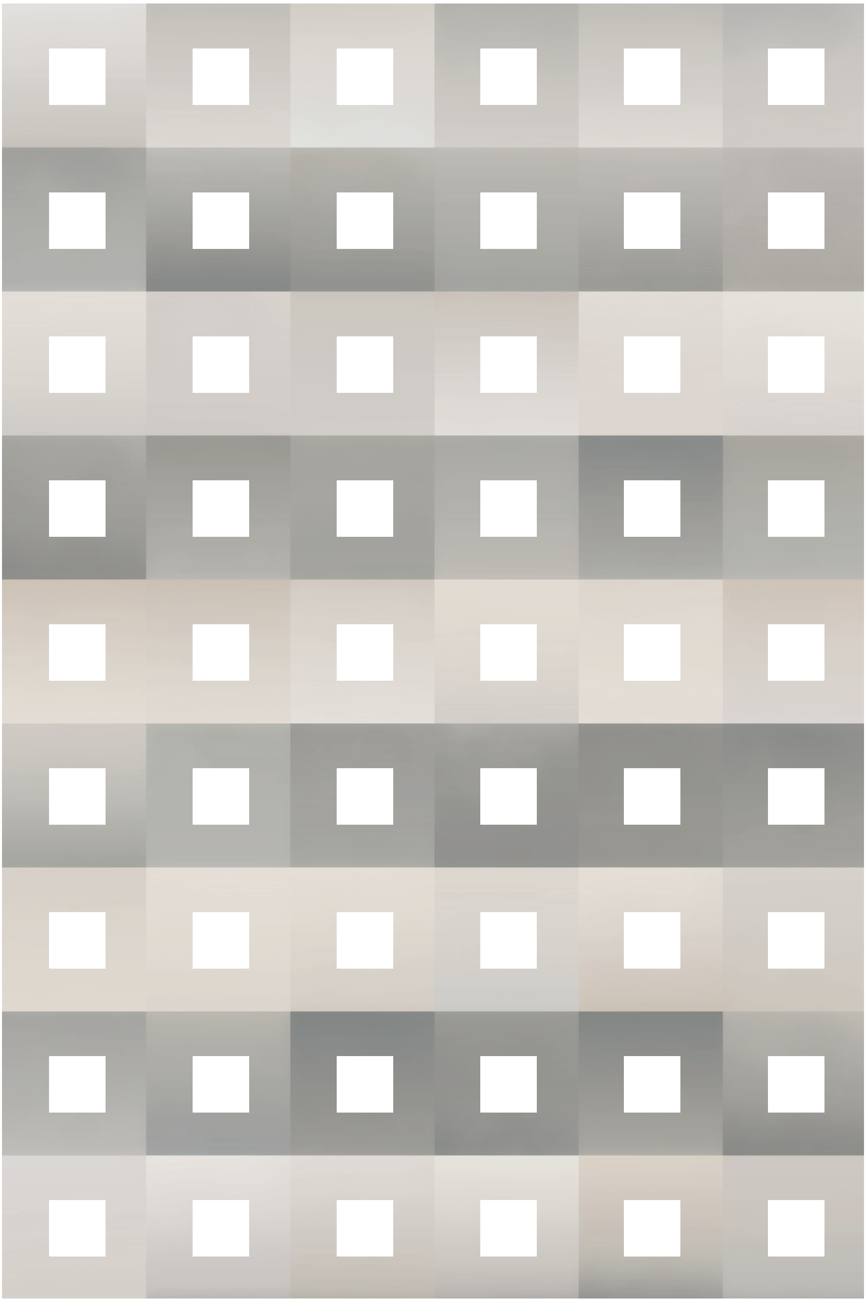
3

EUCLIDEAN TRANSFORMATIONS

In the third part of this book, we will look at Euclidean geometry from a different perspective, that of Euclidean transformations. It is a point of view that has been most closely associated with Felix Klein– that the way to study some property (such as congruence) is to study the maps that preserve it. The first lesson sets the scene with a quick development of analytic geometry. Then it is on to Euclidean isometries– bijections of the Euclidean plane which preserve distance. Over several lessons we will study these isometries, and ultimately we will classify all Euclidean isometries into four types: reflections, rotations, translations, and glide reflections. Then it is time to loosen the restriction a bit to consider bijections which preserve congruence, but not necessarily distance. Finally, we will look at inversion, a type of bijection of the punctured plane (the Euclidean plane minus a point). As luck would have it, inversion provides a convenient bridge into non-Euclidean geometry.

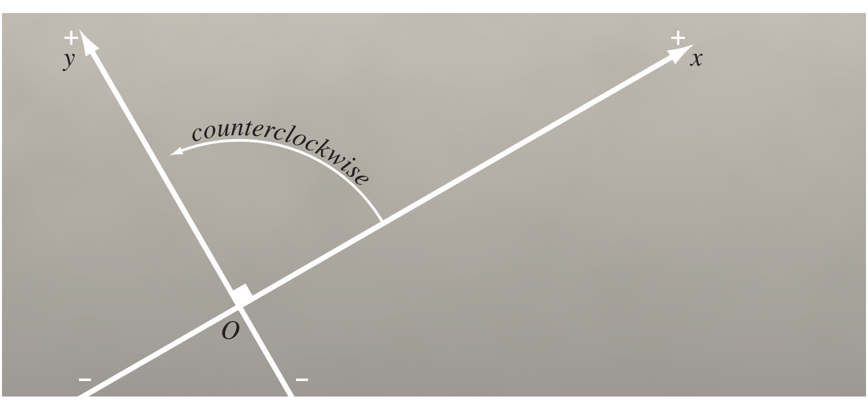


23 BACK ON THE GRID
ANALYTIC GEOMETRY

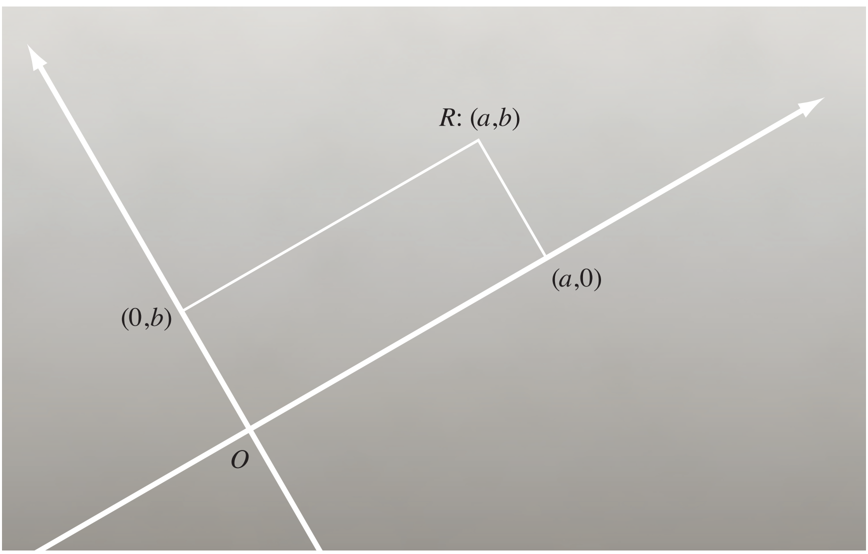
This lesson is just a quick development of analytic geometry and trigonometry in the language of Euclidean geometry. I feel an obligation to provide the connection between traditional Euclidean geometry (as I have developed it in these lessons) and more contemporary analytic geometry, but you should already be comfortable with this material, so feel free to skim through it.

Analytic geometry

At the heart of analytic geometry, there is a correspondence between points and coordinates, ordered pairs of real numbers. The Cartesian approach to that correspondence is a familiar one, but let me quickly run through it. Begin with two perpendicular lines (the choice is arbitrary). These are the x- and y-axes. Their intersection is the origin O. We will want to measure signed distances from O along these axes, and that means we have to assign a positive direction to each axis. From a geometric point of view, the choice of those directions is arbitrary, but there is an established convention as follows. Once directions have been chosen, each axis will be divided into two rays that share O as their common vertex: a positive axis consisting of points whose signed distance from O is positive, and a negative axis consisting of points whose signed distance from O is negative. The convention is that the axes are assigned positive directions so that the positive y-axis is a 90° *counterclockwise* turn from the positive x-axis. Now here's the catch: the geometry itself provides no way to distinguish which direction is the counterclockwise direction. So this is a convention that must be passed along by way of illustrations (and clocks).



A point P on the x-axis is assigned the coordinates $(p,0)$, where p is the signed distance from O to P. A point Q on the y-axis is assigned the coordinates $(0,q)$ where q is the signed distance from O to Q. Most points will not lie on either axis. For these points, we must consider their projections onto the axes. If R is such a point, then we draw the two lines that pass through R and are perpendicular to the two axes. If the points where these perpendiculars cross the axes have coordinates $(a,0)$ and $(0,b)$, then the coordinates of R are (a,b). With this correspondence, every point corresponds to a unique coordinate pair, and every coordinate pair corresponds to a unique point.



The next step is to figure out how to calculate the distance between points in terms of their coordinates. This is pretty much essential for everything else that we are going to do. Let's begin with two special cases.

LEM: VERTICAL DISTANCE

For points that share an x-coordinate, $P_1 = (x, y_1)$ and $P_2 = (x, y_2)$,

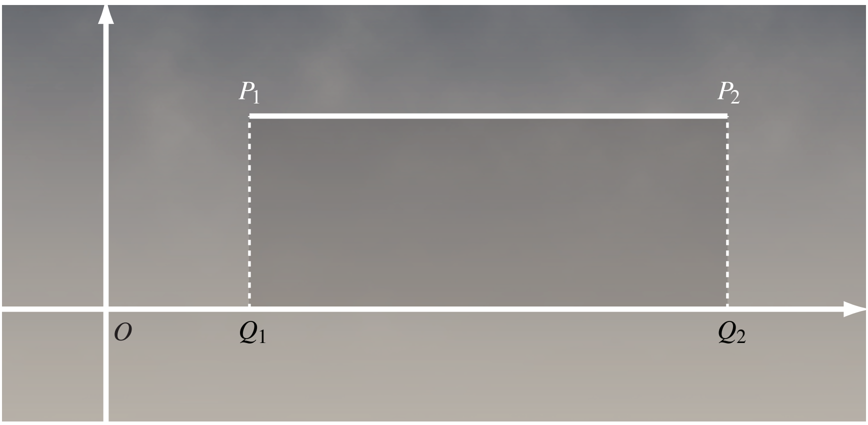
$$|P_1P_2| = |y_1 - y_2|.$$

HORIZONTAL DISTANCE

For points that share a y-coordinate, $P_3 = (x_3, y)$ and $P_4 = (x_4, y)$,

$$|P_3P_4| = |x_3 - x_4|.$$

Proof. I will just prove the first statement. Label two more points, $Q_1 = (0, y_1)$ and $Q_2 = (0, y_2)$. The resulting quadrilateral $P_1P_2Q_2Q_1$ is a rectangle, so its opposite sides P_1P_2 and Q_1Q_2 have to be the same length.



This is where we make the direct connection between coordinates and distance– the coordinates along each axis were chosen to reflect their signed distance from the origin O. To be thorough, though, there are several cases to consider:

$$
\begin{array}{ll}
O * Q_1 * Q_2: & |Q_1Q_2| = |OQ_2| - |OQ_1| = y_2 - y_1 = |y_1 - y_2| \\
O * Q_2 * Q_1: & |Q_1Q_2| = |OQ_1| - |OQ_2| = y_1 - y_2 = |y_1 - y_2| \\
Q_1 * O * Q_2: & |Q_1Q_2| = |OQ_1| + |OQ_2| = -y_1 + y_2 = |y_1 - y_2| \\
Q_2 * O * Q_1: & |Q_1Q_2| = |OQ_2| + |OQ_1| = -y_2 + y_1 = |y_1 - y_2| \\
Q_1 * Q_2 * O: & |Q_1Q_2| = |OQ_1| - |OQ_2| = -y_1 - (-y_2) = |y_1 - y_2| \\
Q_2 * Q_1 * O: & |Q_1Q_2| = |OQ_2| - |OQ_1| = -y_2 - (-y_1) = |y_1 - y_2|
\end{array}
$$

No matter the case, $|P_1P_2| = |Q_1Q_2| = |y_1 - y_2|$. $\square$

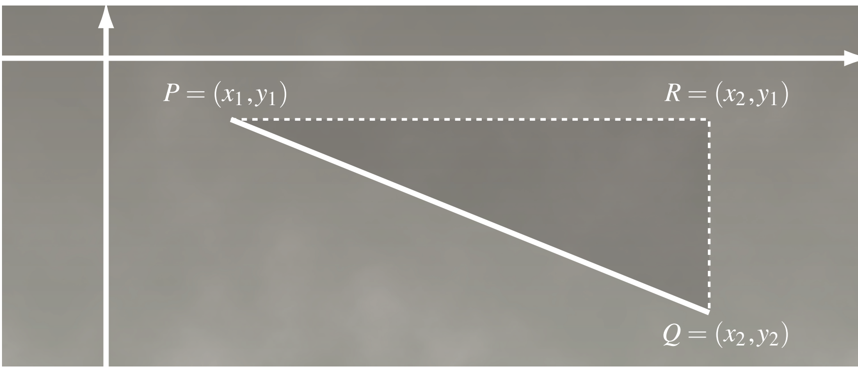
The general distance formula is now an easy consequence of the Pythagorean Theorem.

THM: THE DISTANCE FORMULA

For any two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$,

$$|PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Proof. If P and Q share either x-coordinates or y-coordinates, then this formula reduces down to the special case in the previous lemma (because $\sqrt{a^2} = |a|$). If not, mark one more point: $R = (x_2, y_1)$.



Then $|PR| = |x_1 - x_2|$, and $|RQ| = |y_1 - y_2|$, and $\triangle PRQ$ is a right triangle. By the Pythagorean theorem,

$$|PQ|^2 = |PR|^2 + |QR|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

Now take the square root to get the formula. □

COR: THE EQUATION OF A CIRCLE

The equation of a circle C with center at $P = (h, k)$ and radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

Proof. By definition, the points of C are all those points that are a distance of r from P. Therefore (x, y) is on C if and only if

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

Square both sides of the equation to get the standard form. □

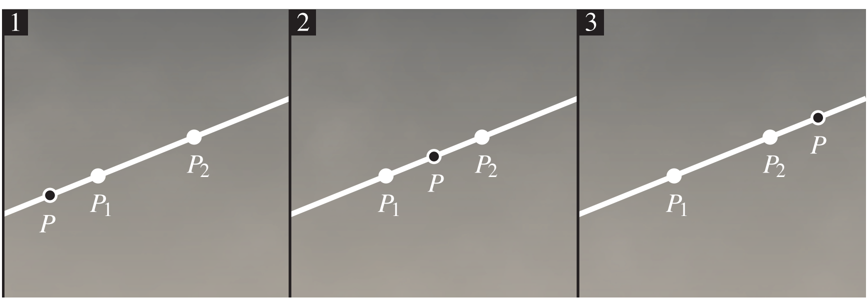
Moving along, lines are next. Intuitively, the key is the idea that a line describes the shortest path between points. That is captured more formally in the triangle inequality, which you should recall states that $|AB|+|BC| \geq |AC|$, but that the equality only happens when $A*B*C$.

PARAMETRIC FORM FOR THE EQUATION OF A LINE
Given two distinct points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on a line ℓ, a third point $P = (x, y)$ lies on ℓ if and only if its coordinates can be written in the form

$$x = x_1 + t(x_2 - x_1) \quad \& \quad y = y_1 + t(y_2 - y_1)$$

for some $t \in \mathbb{R}$.

Proof. The different possible orderings of P, P_1, and P_2 on the line create several scenarios



Let me just take the middle case, where t is between 0 and 1 and P is between P_1 and P_2. It is representative of the other two cases.

$\Longrightarrow$ Show that if $P = (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$ for some value of t between 0 and 1, then P is between P_1 and P_2.

We can directly calculate $|P_1P|$ and $|PP_2|$:

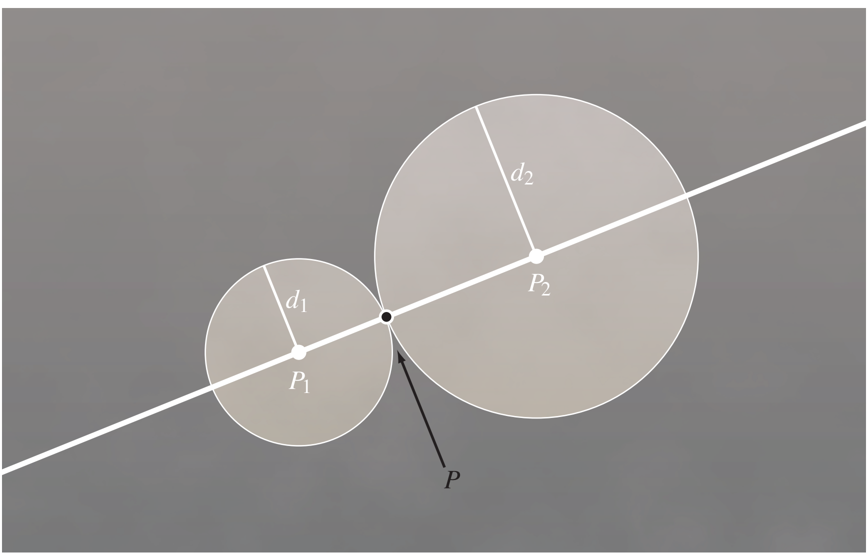
$$\begin{aligned}
|P_1P| &= [(x - x_1)^2 + (y - y_1)^2]^{1/2} \\
&= [(x_1 + t(x_2 - x_1) - x_1)^2 + (y_1 + t(y_2 - y_1) - y_1)^2]^{1/2} \\
&= [(tx_2 - tx_1)^2 + (ty_2 - ty_1)^2]^{1/2} \\
&= t[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2} \\
&= t|P_1P_2|.
\end{aligned}$$

$$\begin{aligned}
|PP_2| &= [(x_2-x)^2+(y_2-y)^2]^{1/2}\\
&= [(x_2-(x_1+t(x_2-x_1)))^2+(y_2-(y_1+t(y_2-y_1)))^2]^{1/2}\\
&= [((1-t)x_2-(1-t)x_1)^2+((1-t)y_2-(1-t)y_1)^2]^{1/2}\\
&= (1-t)[(x_2-x_1)^2+(y_2-y_1)^2]^{1/2}\\
&= (1-t)|P_1P_2|.
\end{aligned}$$

According to the Triangle Inequality, then, P is between P_1 and P_2, since

$$|P_1P|+|PP_2| = t|P_1P_2|+(1-t)|P_1P_2| = |P_1P_2|.$$

$\Longleftarrow$ Show that if P is between P_1 and P_2, then the coordinates of P can be written in the parametric form $(x_1+t(x_2-x_1), y_1+t(y_2-y_1))$ for some value of t between 0 and 1.



Point P is the only point in the plane which is a distance $d_1 = |P_1P|$ from P_1 *and* a distance $d_2 = |PP_2|$ from P_2. Because of that uniqueness, we just need to find a point in parametric form that is also those respective distances from P_1 and P_2. The point that we are looking for is the one where $t = d_1/(d_1+d_2)$. The two calculations, that the distance from this point to P_1 is d_1, and that the distance from this point to P_2 is d_2, are both straightforward, so I will leave them to you. □

From the parametric form it is easy to get to standard form, and from there to point-slope form, slope-intercept form, and so on. The latter steps are standard fare for a pre-calculus course, so I will only go one step further.

STANDARD FORM FOR THE EQUATION OF A LINE
The coordinates (x,y) of the points of a line all satisfy an equation of the form $Ax+By=C$ where A, B, and C are real numbers.

Proof. Suppose that (x_1,y_1) and (x_2,y_2) are distinct points on the line. As we saw in the last theorem, the other points on the line have coordinates (x,y) that satisfy the equations

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1). \end{cases}$$

Now it is just a matter of combining the equations to eliminate the parameter t.

$$\begin{cases} x - x_1 = t(x_2 - x_1) \\ y - y_1 = t(y_2 - y_1). \end{cases}$$

At this point, you could divide the second equation by the first. That eliminates the t variable and also serves as a definition of the slope of a line (in particular, it shows that the slope is constant). But it also presents a potential "divide by zero" scenario, so instead let's multiply:

$$\begin{cases} (x - x_1)(y_2 - y_1) = t(x_2 - x_1)(y_2 - y_1) \\ (y - y_1)(x_2 - x_1) = t(y_2 - y_1)(x_2 - x_1). \end{cases}$$

Set the two equations equal and simplify

$$\begin{aligned} (x - x_1)(y_2 - y_1) &= (y - y_1)(x_2 - x_1) \\ x(y_2 - y_1) - x_1(y_2 - y_1) &= y(x_2 - x_1) - y_1(x_2 - x_1) \\ x(y_2 - y_1) - y(x_2 - x_1) &= x_1(y_2 - y_1) - y_1(x_2 - x_1). \end{aligned}$$

This equation has the proper form, with

$$A = y_2 - y_1 \quad B = -(x_2 - x_1) \quad \& \quad C = x_1(y_2 - y_1) - y_1(x_2 - x_1).$$

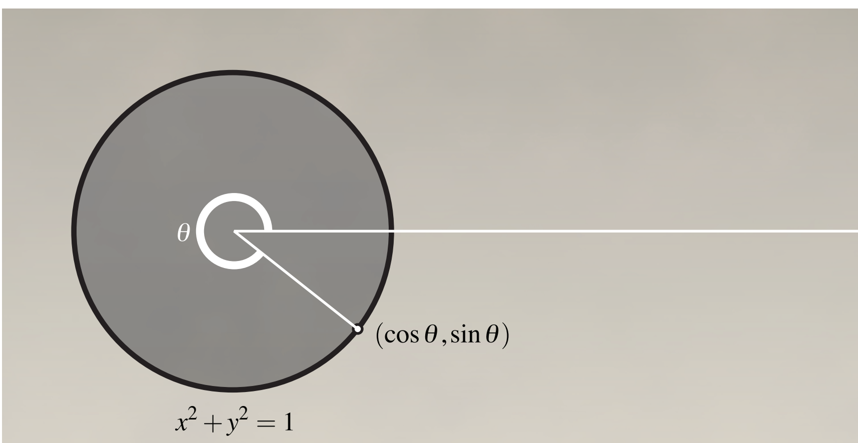
□

Finally, it should be noted that any three real numbers A, B, C do describe a line, so long as A and B are not both zero.

The unit circle approach to trigonometry

At the end of the lesson on similarity, in the exercises, we defined the six trigonometric functions. At that time, we defined them in terms of the angles of a right triangle, which means that they were restricted to values in the interval $(0, \pi/2)$. As you know, there is also a "unit circle approach" that extends these definitions beyond that narrow window. You have seen this before, so I will be as brief as I can be. A point with two *positive* coordinates (x, y) on the unit circle corresponds to a right triangle whose vertices are $(0,0)$, $(x,0)$ and (x,y). If θ is the measure of the angle at the origin, then $\cos\theta = x$ and $\sin\theta = y$ (because the hypotenuse has length one). Now just continue that: any ray from the origin forms an angle θ measured in the counterclockwise direction from the x-axis. That ray intersects the unit circle at a point (x, y) and we define

$$\cos(\theta) = x \quad \sin(\theta) = y.$$

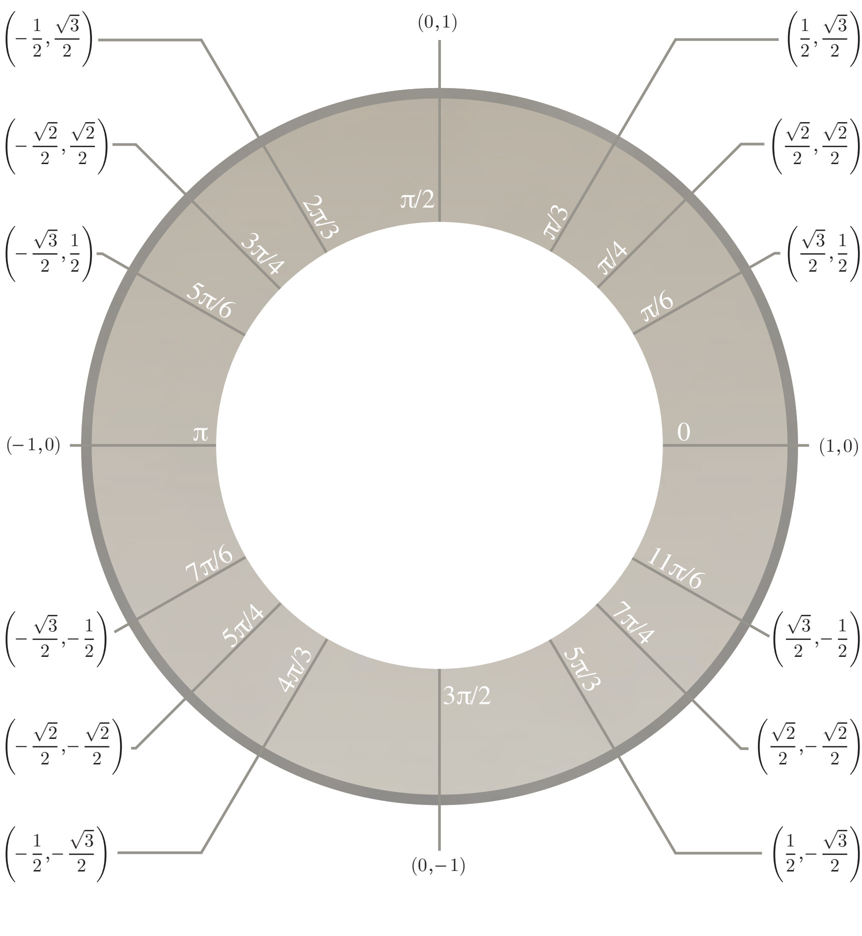


Allowing for both proper and reflex angles, that extends the domains of sine and cosine to $[0, 2\pi)$, but we can go farther. Informally, we need to allow the ray to spin around the circle more than once (for θ values greater than 2π) or in the counterclockwise direction (for negative θ). Formally, this can be done by imposing periodicity:

$$\cos(\theta + 2n\pi) = \cos(\theta) \quad \sin(\theta + 2n\pi) = \sin(\theta) \quad \forall n \in \mathbb{N}.$$

Use an isosceles and equilateral triangle to find sine and cosine values for π/3, π/4, and π/6. Use the symmetry of the circle to extend outside of quadrant I.

$x^2 + x^2 = 1 \implies x = \sqrt{2}/2$ $\qquad$ $(1/2)^2 + y^2 = 1 \implies y = \sqrt{3}/2$



The other four trigonometric functions (tangent, cotangent, secant, cosecant) are defined similarly as the ratios

$$\tan(\theta) = y/x \quad \cot(\theta) = x/y \quad \sec(\theta) = 1/x \quad \csc(\theta) = 1/y.$$

There are a lot of relationships between the trigonometric functions, some easy and some subtle. Let's get the easy ones out of the way. From the very definitions of the functions, we get the reciprocal identities

$$\sec\theta = \frac{1}{\cos\theta} \quad \csc\theta = \frac{1}{\sin\theta} \quad \cot\theta = \frac{1}{\tan\theta},$$

and identities that relate tangent and cotangent to sine and cosine

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

From the equation of the circle $x^2 + y^2 = 1$, we get the Pythagorean identities:

$$\sin^2\theta + \cos^2\theta = 1 \quad \tan^2\theta + 1 = \sec^2\theta \quad 1 + \cot^2\theta = \csc^2\theta.$$

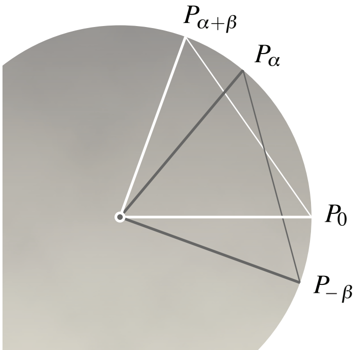
By comparing angles taken in the counterclockwise and clockwise directions, we see that cosine and secant are even functions (where $f(-x) = f(x)$) and that the other four are odd functions (where $f(-x) = -f(x)$).

Beyond these, there is a second tier of identities– double angle, half angle, power reduction, etc – that are not so immediately clear. They can all be derived from two big identities, the addition formulas for sine and cosine, but the proofs of those two formulas require a more careful look at the geometry of the unit circle. To close out this lesson, I will prove the two addition formulas.

ADDITION RULE FOR COSINE

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Proof. The key to the proof is to compare two distances which we know to be the same– one distance expressed in terms of the angle $\alpha + \beta$, the other in terms of the individual angles α and β. The real trick to this is to make the right choice of distances. In particular, you have to be careful so that you don't get stuck with a $\sin(\alpha+\beta)$ term in the first calculation. On the unit circle, label the following points:



$$\begin{aligned}
P_0 &= (1,0)\\
P_\alpha &= (\cos\alpha, \sin\alpha)\\
P_{-\beta} &= (\cos(-\beta), \sin(-\beta))\\
&= (\cos\beta, -\sin\beta)\\
P_{\alpha+\beta} &= (\cos(\alpha+\beta), \sin(\alpha+\beta))
\end{aligned}$$

If O is the origin, then the triangles $\triangle OP_0P_{\alpha+\beta}$ and $\triangle OP_{-\beta}P_\alpha$ are congruent (S·A·S: in each triangle, two of the sides are radii, and the angle between them measures $\alpha+\beta$). That means that the two segments $P_0P_{\alpha+\beta}$ and $P_{-\beta}P_\alpha$ have to be congruent, and so we can compare their lengths (it is actually easier to work with the squares of those lengths). Throughout these calculations, we make repeated use of the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned}
|P_0P_{\alpha+\beta}|^2 &= (\cos(\alpha+\beta)-1)^2 + (\sin(\alpha+\beta)-0)^2\\
&= \cos(\alpha+\beta)^2 - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta)\\
&= 2 - 2\cos(\alpha+\beta).
\end{aligned}$$

$$\begin{aligned}
|P_{-\beta}P_\alpha|^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha + \sin\beta)^2\\
&= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta\\
&\qquad + \sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta\\
&= 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta.
\end{aligned}$$

Set these two expressions equal to each other, subtract 2 and divide by -2 to get the desired formula

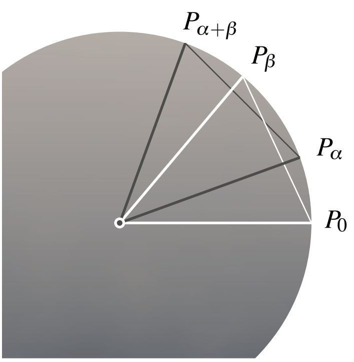
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

□

ADDITION RULE FOR SINE

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Proof. For this proof, one approach would be to use the cofunction identity $\sin(x) = \cos(\pi/2 - x)$ followed by the addition rule for cosine that we just derived. That is pretty easy, but you would have to verify the cofunction identity first. That too is easy for x values between 0 and $\pi/2$, but gets to be a nuisance once you have to consider all the other possible values of x. I think it is easier to do something like the last proof– compare some distances and then do a little algebra. On the unit circle, label the following points



$$\begin{aligned}
P_0 &= (1,0)\\
P_\alpha &= (\cos\alpha, \sin\alpha)\\
P_\beta &= (\cos\beta, \sin\beta)\\
P_{\alpha+\beta} &= (\cos(\alpha+\beta), \sin(\alpha+\beta)).
\end{aligned}$$

By S·A·S, the segments $P_\alpha P_{\alpha+\beta}$ and P_0P_β are congruent. Let's compare those two distances. Here we go (note the use of the addition rule for cosine midway through the first distance calculation).

$$\begin{aligned}
|P_\alpha P_{\alpha+\beta}|^2 &= (\cos(\alpha+\beta) - \cos(\alpha))^2 + (\sin(\alpha+\beta) - \sin(\alpha))^2\\
&= \cos^2(\alpha+\beta) - 2\cos\alpha\cos(\alpha+\beta) + \cos^2\alpha\\
&\qquad + \sin^2(\alpha+\beta) - 2\sin\alpha\sin(\alpha+\beta) + \sin^2\alpha\\
&= 2 - 2\cos\alpha\cos(\alpha+\beta) - 2\sin\alpha\sin(\alpha+\beta)\\
&= 2 - 2\cos\alpha(\cos\alpha\cos\beta - \sin\alpha\sin\beta) - 2\sin\alpha\sin(\alpha+\beta)\\
&= 2 - 2\cos^2\alpha\cos\beta + 2\sin\alpha\cos\alpha\sin\beta - 2\sin\alpha\sin(\alpha+\beta)
\end{aligned}$$

and

$$\begin{aligned}|P_0P_\beta|^2 &= (\cos\beta - 1)^2 + (\sin\beta - 0)^2\\ &= \cos^2\beta - 2\cos\beta + 1 + \sin^2\beta\\ &= 2 - 2\cos\beta.\end{aligned}$$

Now set these two expressions equal, subtract 2 from both sides and divide through by -2 to get

$$\cos^2\alpha\cos\beta - \sin\alpha\cos\alpha\sin\beta + \sin\alpha\sin(\alpha+\beta) = \cos\beta.$$

In this equation solve for the $\sin(\alpha+\beta)$ term

$$\begin{aligned}\sin\alpha\sin(\alpha+\beta) &= \cos\beta - \cos^2\alpha\cos\beta + \sin\alpha\cos\alpha\sin\beta\\ &= \cos\beta(1-\cos^2\alpha) + \sin\alpha\cos\alpha\sin\beta\\ &= \cos\beta\sin^2\alpha + \sin\alpha\cos\alpha\sin\beta\\ &= \sin\alpha(\sin\alpha\cos\beta + \cos\alpha\sin\beta).\end{aligned}$$

As long as $\sin\alpha$ is not zero, we can divide both sides by that, and what's left over is what we want. What if $\sin\alpha$ is zero? Well, that happens when α is any multiple of π, and those cases are easy enough to handle on their own. On the left side, adding $n\pi$ corresponds to a half-turn or a whole turn around the unit circle, so

$$\sin(n\pi+\beta) = \begin{cases}\sin\beta & \text{if } n \text{ is even}\\ -\sin\beta & \text{if } n \text{ is odd.}\end{cases}$$

Compare that to the right side

$$\begin{aligned}\sin(n\pi)\cos\beta + \cos(n\pi)\sin\beta &= 0\cdot\cos\beta + \cos(n\pi)\sin\beta\\ &= \begin{cases}\sin\beta & \text{if } n \text{ is even}\\ -\sin\beta & \text{if } n \text{ is odd}\end{cases}\end{aligned}$$

They are the same. $\square$

Exercises

1. Prove the midpoint formula. Let $P = (a,b)$ and $Q = (c,d)$. Verify that the coordinates of the midpoint of PQ are
$$\left(\frac{a+c}{2}, \frac{b+d}{2}\right).$$

2. Show that the points on the circle with center (h,k) and radius r can be described by the parametric equations
$$\begin{cases} x(\theta) = h + r\cos\theta \\ y(\theta) = k + r\sin\theta \end{cases}.$$

3. Let ℓ_1 and ℓ_2 be perpendicular lines, neither of which is a vertical line. Show that the slopes of ℓ_1 and ℓ_2 are negative reciprocals of one another.

4. Verify that the triangle with vertices at $(0,0)$, $(2a,0)$, and $(a, a\sqrt{3})$ is equilateral.

5. Find the equation of the circle which passes through the three points: $(0,0)$, $(4,2)$ and $(2,6)$.

6. Let $\triangle ABC$ be the triangle with vertices at the coordinates $A = (0,0)$, $B = (1,0)$, $C = (a,b)$. Find the coordinates of its circumcenter, orthocenter, and centroid (in terms of a and b).

7. All of the special values on the unit circle can be written in the form $n\pi/12$, but not all values of that form are represented. Find the coordinates on the unit circle for the angles $\theta = \pi/12, 5\pi/12, 7\pi/12$, and $11\pi/12$.

   The remaining exercises verify some common trigonometric identities that we will need to for later calculations. You don't need to do them all– I really just want to have all of these identities together in one place.

8. Use the addition formulas to derive the cofunction identities.

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

$$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$$

$$\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta \qquad \csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta$$

9. Use the addition formulas to derive the double angle formulas

$$\sin(2\theta)=2\sin\theta\cos\theta$$

$$\begin{aligned}\cos(2\theta)&=\cos^2\theta-\sin^2\theta\\&=2\cos^2\theta-1\\&=1-2\sin^2\theta\end{aligned}$$

$$\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$$

10. Use the double angle formulas for cosine to derive the power-reduction formulas

$$\sin^2\theta=\frac{1-\cos(2\theta)}{2}$$

$$\cos^2\theta=\frac{1+\cos(2\theta)}{2}$$

$$\tan^2\theta=\frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

11. Use the power-reduction formulas to derive the half-angle formulas

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}=\frac{\sin\theta}{1+\cos\theta}$$

12. Verify the product-to-sum formulas

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

13. Verify the sum-to-product formulas

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha-\sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$