24 ISOMETRIES

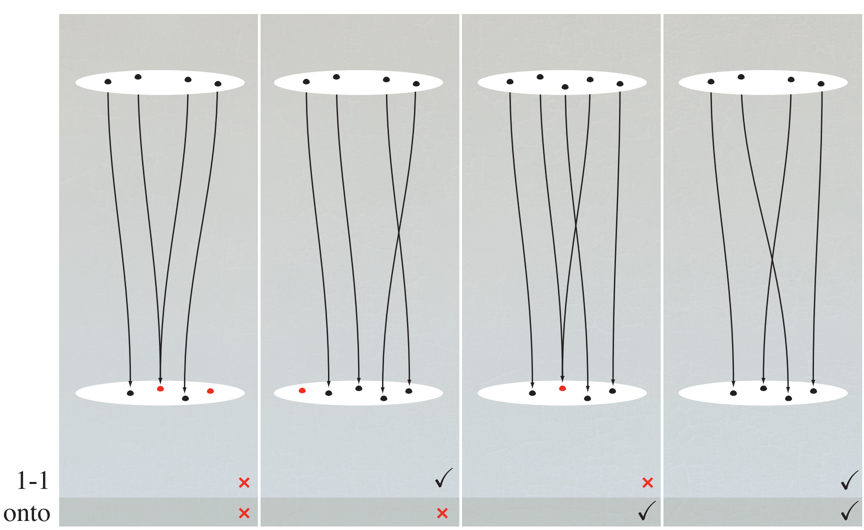
BOW TO YOUR PARTNER. BOW TO YOUR CORNER.

One of the prevailing philosophies of modern mathematics is that in order to study something, you need to study the types of maps that preserve it– that is, the types of maps that leave it invariant. For instance, in group theory we study group homomorphisms because they preserve the group operation (in the sense that $f(a \cdot b) = f(a) \cdot f(b)$). In Euclidean geometry there are several structures that might be worth preserving– incidence, order, congruence– but in the next few lessons our focus will be on mappings that preserve distance.

Definitions

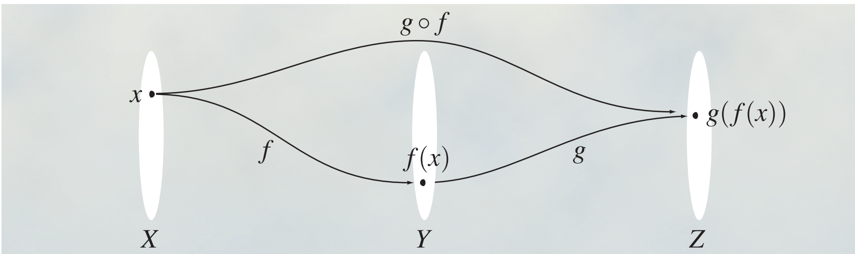
Let's start with a review of some basic terminology associated with maps from one set to another.

DEF: ONE-TO-ONE, ONTO, AND BIJECTIVE MAPPINGS
A map $f : X \to Y$ is:
· *one-to-one* if $f(x) = f(y) \implies x = y$;
· *onto* if for every $y \in Y$ there is an $x \in X$ such that $f(x) = y$;
· *bijective* if it is both one-to-one and onto.



Under the right circumstances, two mappings may be chained together: the composition of $f : X \to Y$ and $g : Y \to Z$ is

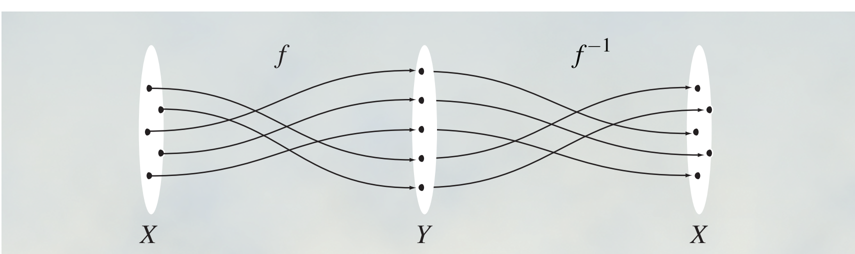
$$g \circ f : X \to Z : g \circ f(x) = g(f(x)).$$



This type of composition is usually not commutative– in fact, $f \circ g$ may not even be defined. It is associative, though, and that is a very essential property. For any space X the map

$$id : X \to X : id(x) = x$$

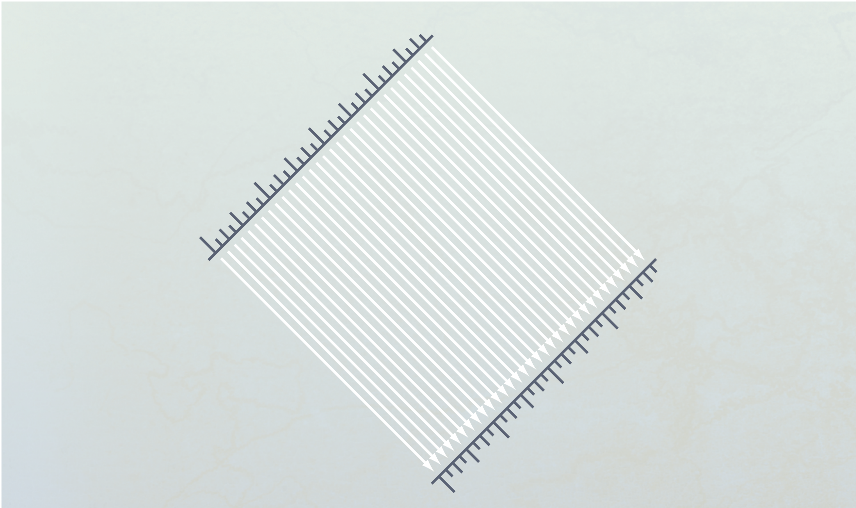
is called the identity map. Two maps $f : X \to Y$ and $g : Y \to X$ are inverses of one another if $f \circ g$ is the identity map on Y and $g \circ f$ is the identity map on X. In order for a map to have an inverse, it must be bijective (and conversely, any bijection is invertible).



DEF: AUTOMORPHISM

An automorphism is a bijective mapping f from a space to itself.

We are interested in automorphisms of the Euclidean plane, but not just any automorphisms. We want the ones that do not distort the distances between points. These are called Euclidean isometries.



DEF: ISOMETRY
Let $\mathbb{E}$ denote the set of points of the Euclidean plane. A Euclidean isometry is an automorphism $f : \mathbb{E} \to \mathbb{E}$ that preserves the distance between points: for all A, B in $\mathbb{E}$, $|f(A)f(B)| = |AB|$.

I will leave the proof of the following basic properties of isometries to you. If you are familiar with the concept of a group, these properties mean that the set of Euclidean isometries is a group.

LEM: BASIC PROPERTIES OF ISOMETRIES
The composition of two isometries is an isometry. The identity map is an isometry. The inverse of an isometry is an isometry.

Recall that everything we have done in Euclidean geometry floats on five undefined terms: point, line, on, between, and congruence. An isometry is defined in terms of its behavior on points, but the distance preservation condition has implications for the remaining undefined terms as well.

LEM: ISOMETRIES AND CONGRUENCE
An isometry preserves both segment and angle congruence. That is,

$$AB \simeq A'B' \implies f(A)f(B) \simeq f(A')f(B')$$
$$\angle ABC \simeq \angle A'B'C' \implies \angle f(A)f(B)f(C) \simeq \angle f(A')f(B')f(C')$$

Proof. The segment congruence part is easy, because isometries preserve distance and hence segment length, and it is those lengths that determine whether or not segments are congruent: if $AB \simeq A'B'$, then

$$|f(A)f(B)| = |AB| = |A'B'| = |f(A')f(B')|$$

and so $f(A)f(B) \simeq f(A')f(B')$. The angle congruence part is not that hard either, but we will need to use a few of the triangle congruence theorems. Relocate, if necessary, A' and C' on their respective rays so that $BA \simeq B'A'$ and $BC \simeq B'C'$. By S·A·S, the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent. The corresponding sides of these two triangles are congruent, and from the first part of the proof, the congruences are transferred by f:

$$\begin{aligned} AB \simeq A'B' &\implies f(A)f(B) \simeq f(A')f(B') \\ BC \simeq B'C' &\implies f(B)f(C) \simeq f(B')f(C') \\ CA \simeq C'A' &\implies f(C)f(A) \simeq f(C')f(A') \end{aligned}$$

By S·S·S, triangles $\triangle f(A)f(B)f(C)$ and $\triangle f(A)f(B)f(C)$ are congruent, and so the corresponding angles $\angle f(A)f(B)f(C)$ and $\angle f(A')f(B')f(C')$ are congruent. $\square$

If you were paying attention in the last proof, you may have noticed that it could easily be tweaked to say a bit more: an isometry doesn't preserve just distance– it also preserves angle measure, in the sense that

$$(\angle ABC) = (\angle f(A)f(B)f(C)).$$

This is useful. In fact, we will use it in the last proof of this lesson.

LEM: ISOMETRIES, INCIDENCE AND ORDER

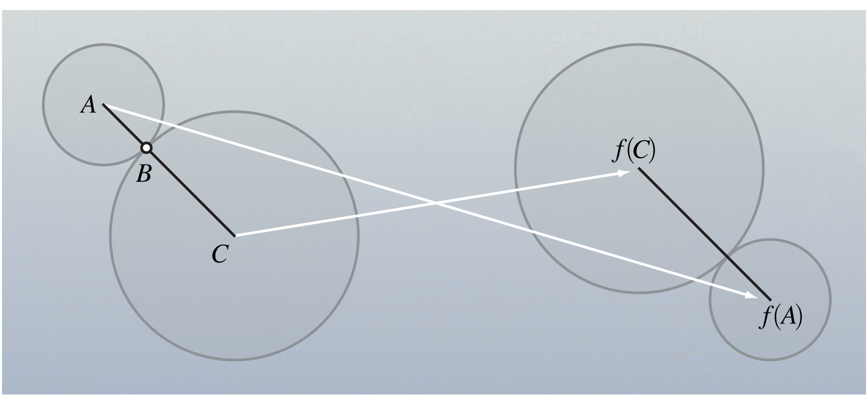
If A, B, and C are collinear, in the order $A*B*C$, and f is an isometry, then $f(A)$, $f(B)$, and $f(C)$ are collinear, in the order $f(A)*f(B)*f(C)$.

Proof. Suppose $A*B*C$. Then, by segment addition

$$|AC| = |AB| + |BC|.$$

Distance is invariant under f, so we can make the substitutions

$$|f(A)f(B)| = |AB|, \quad |f(B)f(C)| = |BC|, \quad |f(A)f(C)| = |AC|,$$



to get

$$|f(A)f(C)| = |f(A)f(B)| + |f(B)f(C)|.$$

This is the degenerate case of the Triangle Inequality: the only way this equation can be true is if $f(A)$, $f(B)$, and $f(C)$ are collinear, and that $f(B)$ is between $f(A)$ and $f(C)$. $\square$

In the last result we were talking about three points, but by extension, this means that all the points on a line are mapped again to collinear points. In other words, an isometry, which is defined as a bijection of points, is also a bijection of the lines of the geometry. Further, an isometry maps segments to segments, rays to rays, angles to angles, and circles to circles. Well, here's an opportunity to simplify notation. When I apply an isometry f to a segment AB, for example, instead of writing $f(A)f(B)$, I will go with the more streamlined $f(AB)$. For an angle $\angle ABC$, instead of $\angle f(A)f(B)f(C)$, I will write $f(\angle ABC)$. And so on.

Fixed points

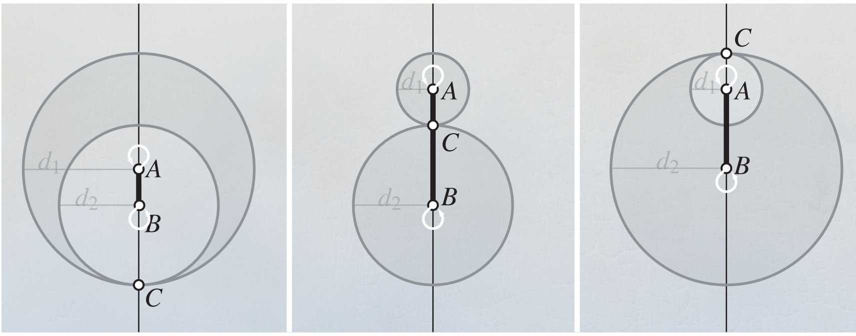
The overarching goal of the next few lessons is to classify all Euclidean isometries. It turns out that one of the keys to this is *fixed points*.

> DEF: FIXED POINT
> A point P is a fixed point of an isometry f if $f(P) = P$.

The first big step towards a classification is to answer the following question:

> Given isometries f_1 and f_2, which may be described in very different ways, how do we figure out if they are really the same?

Showing that they are *not* the same is usually easy– you just need to find one point P where $f_1(P) \neq f_2(P)$. Showing that they *are* the same seems like a more difficult task. At the most basic level, isometries are functions of the Euclidean plane. Without any additional structure, the only way to show two functions are equal is to show that they agree on the value of all points. This is because the behavior of an arbitrary function is quite unconstrained. Fortunately, the bijection and distance-preserving properties of an isometry impose significant constraints on its behavior. Those constraints mean that we can determine whether or not two isometries are the same by looking at just a few points.



f(C) must still be on both of these circles.

THM: TWO FIXED POINTS

If an isometry f fixes two distinct points A and B, then it fixes all the points of the line $\leftarrow AB \rightarrow$.

Proof. Let C be a third point on this line. Label its distances from A as d_1 and from B as d_2. The key here is that C is the only point that is a distance d_1 from A *and* a distance d_2 from B (I think this is intuitively clear, but for a more formal point of view, you can look back at our investigation of the possible intersections of circles in Lesson 16). Now hit these three points with the isometry f. Distances stay the same, so $f(C)$ is still a distance d_1 from $f(A) = A$, and $f(C)$ is still a distance d_2 from $f(B) = B$. That means that $f(C)$ must be C. □

THM: THREE (NON-COLLINEAR) FIXED POINTS

If an isometry f fixes three non-collinear points A, B, and C, then it fixes all points (it is the identity isometry).

Proof. By the last result, f must fix all the points on each of the lines $\leftarrow AB \rightarrow$, $\leftarrow AC \rightarrow$, and $\leftarrow BC \rightarrow$. Now suppose that D is a point that is not on any of those lines. We need to show that D is a fixed point as well. Choose a point M that is between A and B. It is fixed by f. According to Pasch's lemma, the line $\leftarrow DM \rightarrow$ must intersect at least one other side of $\triangle ABC$. Call this intersection N. It too is fixed by f. Therefore D is on a line $\leftarrow MN \rightarrow$ with two fixed points. According to the previous result, it is a fixed point. □

A line through D intersecting two fixed lines.

Now we can answer the question I posed at the start of this section: how much do we need to know about two isometries before we can say they are the same?

THM: THREE NON-COLLINEAR POINTS ARE ENOUGH

If two isometries f_1 and f_2 agree on three non-collinear points, then they are equal.

Proof. Suppose that A, B, and C are three non-collinear points, and that

$$f_1(A) = f_2(A) \quad f_1(B) = f_2(B) \quad f_1(C) = f_2(C).$$

Applying f_2^{-1} to both sides of each of these equations,

$$\begin{aligned} f_2^{-1} \circ f_1(A) &= f_2^{-1} \circ f_2(A) = id(A) = A, \\ f_2^{-1} \circ f_1(B) &= f_2^{-1} \circ f_2(B) = id(B) = B, \\ f_2^{-1} \circ f_1(C) &= f_2^{-1} \circ f_2(C) = id(C) = C. \end{aligned}$$

Therefore $f_2^{-1} \circ f_1$ has three non-collinear fixed points– it must be the identity, and so

$$\begin{aligned} & f_2^{-1} \circ f_1 = id \\ & f_2 \circ f_2^{-1} \circ f_1 = f_2 \circ id \\ & id \circ f_1 = f_2 \\ & f_1 = f_2. \end{aligned}$$

$\square$

The analytic viewpoint

To wrap up this lesson, let's look at isometries from the analytical point of view. Any isometry defines a function on the coordinate pairs. As we have seen, isometries themselves are fairly structured, so it makes sense, then, that the functions they define on the coordinate pairs would have to be similarly inflexible. That is indeed the case.

GENERAL FORM FOR AN ISOMETRY
Any Euclidean isometry T has analytic equations that can be written in one of two matrix forms

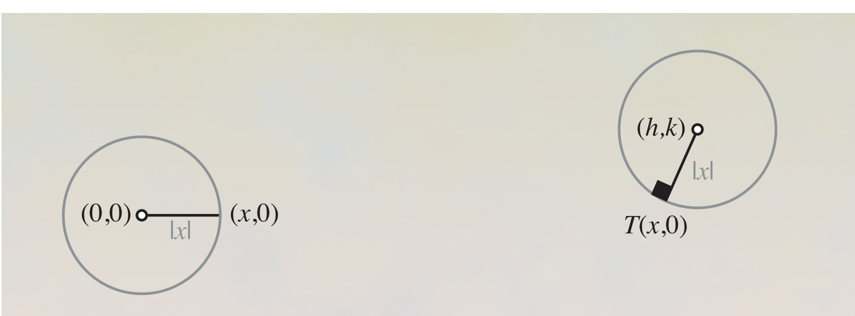
$$(1) \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$(2) \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

where h, k, and θ are real numbers.

Proof. Let T be an isometry. Ultimately, we want to know about $T(x,y)$, but it will take a few steps to get there, starting with the origin, moving to the point $(x,0)$, and then finally to (x,y).

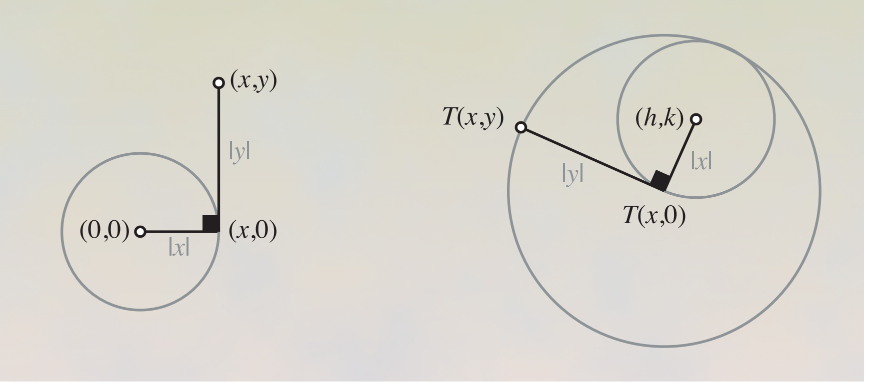
The origin $(0,0)$. This is the easy one. Since the origin is our first point of consideration, there are no limitations on where it goes (we don't know it yet, but there are isometries that take any point to any other point of the plane). Set h and k by looking at what happens to the origin: set $(h,k) = T(0,0)$.



The point $(x,0)$. An isometry preserves distances, and the distance from $(x,0)$ to the origin is $|x|$. Applying the isometry to both of those points, the distance from $T(x,0)$ to (h,k) also has to be $|x|$. In other words, $T(x,0)$ is on the circle with center (h,k) and radius $|x|$. If you did the exercise in the last lesson on parametrizing circles (or if you have worked with parametrized circles in calculus), then you know this means that $T(x,0)$ has to have the form

$$(h+|x|\cos\theta,\ k+|x|\sin\theta)$$

for some value of θ. In fact (and I will leave it to you to figure out why), the absolute value signs around the x are not needed.



The point (x,y). Likewise, since the distance from $(x,0)$ to (x,y) is $|y|$, $T(x,y)$ has to be on the circle centered at $T(x,0)$ with radius $|y|$. That means its coordinates can be written in the form

$$(h+x\cos\theta+|y|\cos\phi,\ k+x\sin\theta+|y|\sin\phi)$$

for some value of ϕ. The possibilities are more limited than that, though: the three points $(0,0)$, $(x,0)$ and (x,y) form a right angle at $(x,0)$. Since an isometry preserves angle measures, the images of these three points must also form a right angle. This can only happen if $\phi = \theta+\pi/2$ or $\phi = \theta-\phi/2$. As before, the absolute value signs around the y can be dropped and that gets us to:

$$\left(h+x\cos\theta+y\cos\left(\theta\pm\frac{\pi}{2}\right),\ k+x\sin\theta+y\sin\left(\theta\pm\frac{\pi}{2}\right)\right).$$

Now use the addition formulas for sine and cosine

$$\cos(\theta \pm \pi/2) = \cos\theta\cos(\pm\pi/2) - \sin\theta\sin(\pm\pi/2) = \mp\sin\theta$$
$$\sin(\theta \pm \pi/2) = \sin\theta\cos(\pm\pi/2) + \cos\theta\sin(\pm\pi/2) = \pm\cos\theta$$

and the coordinates for $T(x,y)$ take on the form

$$(1)\quad T(x,y) = (h + x\cos\theta - y\sin\theta, k + x\sin\theta + y\cos\theta)$$
$$(2)\quad T(x,y) = (h + x\cos\theta + y\sin\theta, k + x\sin\theta - y\cos\theta).$$

Written in matrix form, these are

$$(1)\quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$(2)\quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

□

Exercises

1. Let T be an isometry and let r be a ray with endpoint O. Prove that $T(r)$ is also a ray, with endpoint $T(0)$.

2. Verify that if ℓ_1 and ℓ_2 are parallel lines and T is an isometry, then $T(\ell_1)$ and $T(\ell_2)$ will be parallel.

3. Let T be an isometry and let A and B be two points that are on the same side of a line ℓ. Prove that $T(A)$ and $T(B)$ are on the same side of $T(\ell)$.

4. Let T be an isometry and let D be a point in the interior of angle $\angle ABC$. Prove that $T(D)$ is a point in the interior of $T(\angle ABC)$.

5. Let M be the midpoint of a segment AB, and let T be an isometry so that $T(A) = B$ and $T(B) = A$. Prove that M is a fixed point of this isometry.

6. Given a proper angle $\angle ABC$ and an isometry T such that

$$(1)\ T(BA\!\rightarrow) = BC\!\rightarrow \quad \& \quad (2)\ T(BC\!\rightarrow) = BA\!\rightarrow,$$

show that T fixes all the points of the angle bisector of $\angle ABC$.

7. In the final theorem of this lesson I showed that every isometry can be written in one of two forms. Prove the converse, that any mapping of that form is an isometry.