4. CONGRUENCE VERSE II
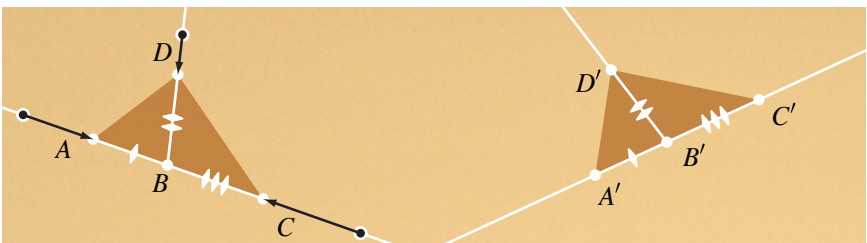## OBJECTIVE: AAS

The ultimate objective of this lesson is derive a third triangle congruence theorem, A·A·S. The basic technique I used in the last chapter to prove S·A·S and A·S·A does not quite work this time though, so along the way we are going to get to see a few more of the tools of neutral geometry: supplementary angles, the Alternate Interior Angle Theorem, and the Exterior Angle Theorem.

Supplementary Angles

There aren't that many letters in the alphabet, so it is easy to burn through most of them in a single proof if you aren't frugal. Even if your variables don't run the full gamut from A to Z, it can be a little challenging just trying to keep up with them. Some of this notation just can't be avoided; fortunately, some of it can. One technique I like to use to cut down on some notation is what I call "relocation". Let's say you are working with a ray $AB \rightarrow$. Now you can't change the endpoint A without changing the ray itself, but there is a little flexibility with the point B. If B' is any other point on the ray (other than A), then $AB \rightarrow$ and $AB' \rightarrow$ are actually the same. So rather than introduce a whole new point on the ray, I like to just "relocate" B to a more convenient location. The same kind of technique can also be used for angles and lines. Let me warn you: you must be careful not to abuse this relocation power. I have seen students relocate a point to one intersection, use the fact that the point is at that intersection in their proof, and then relocate it again a few steps later to another location. That is obviously bad! Yes there is some flexibility to the placement of some of these points, but once you have used up that flexibility, the point has to stay put.
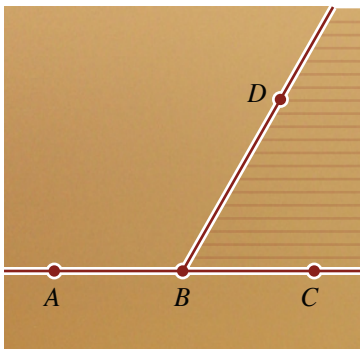

Relocation of points is a shortcut to cut down on notation. Illustrated here are the relocations of points A, B, and C to make the congruences needed for the proof that the supplements of congruent angles are congruent.

Three noncolinear points A, B, and C define an angle $\angle ABC$. When they are colinear, they do not define a proper angle, but you may want to think of them as forming a kind of degenerate angle. If $A * B * C$, then A, B, and C form what is called a "straight angle". One of the most basic relationships that two angles can have is defined in terms of these straight angles.

DEF: SUPPLEMENTARY ANGLES
Suppose that A, B and C form a straight angle with $A * B * C$. Let D be a fourth point which is not the line through A, B and C. Then $\angle ABD$ and $\angle CBD$ are *supplementary angles*.

A pair of supplementary angles:
$\angle ABD$ *and* $\angle DBC$

Supplements have a nice and healthy relationship with congruence as related in the next theorem.

THM: CONGRUENT SUPPLEMENTS
The supplements of congruent angles are congruent: given two pairs of supplementary angles
Pair 1: $\angle ABD$ and $\angle CBD$ and
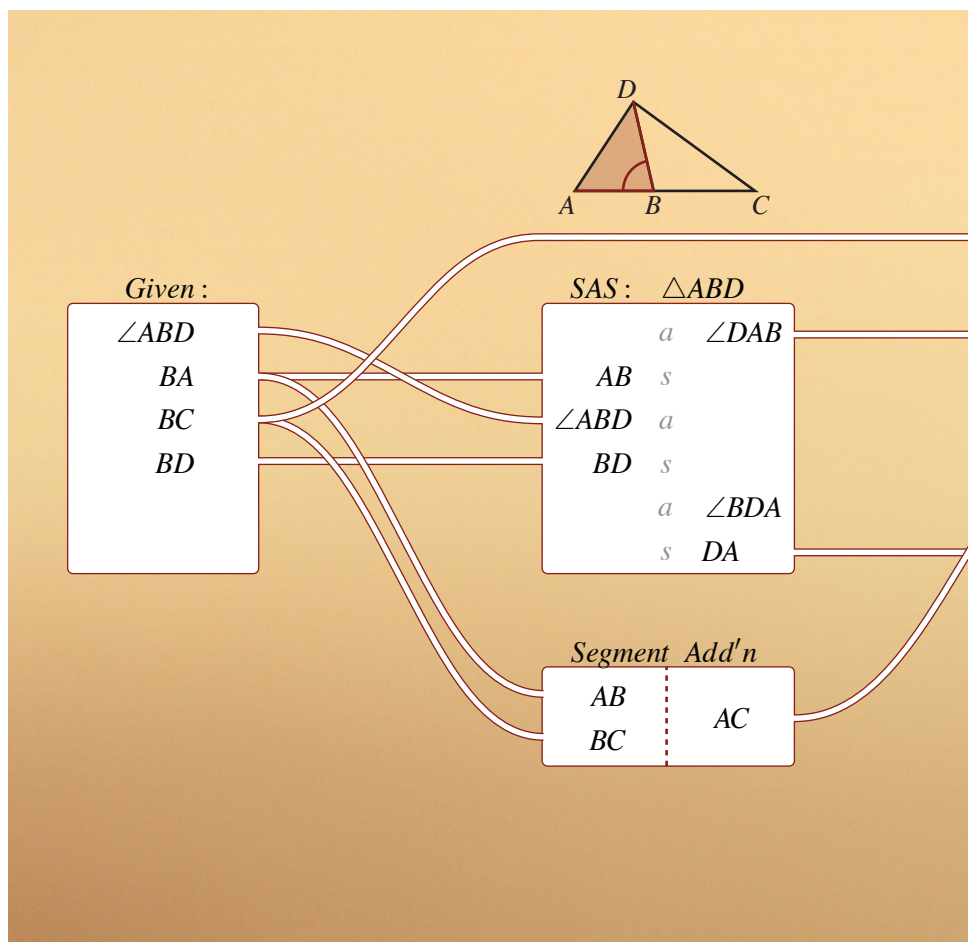*Pair 2:* $\angle A'B'D'$ and $\angle C'B'D'$,
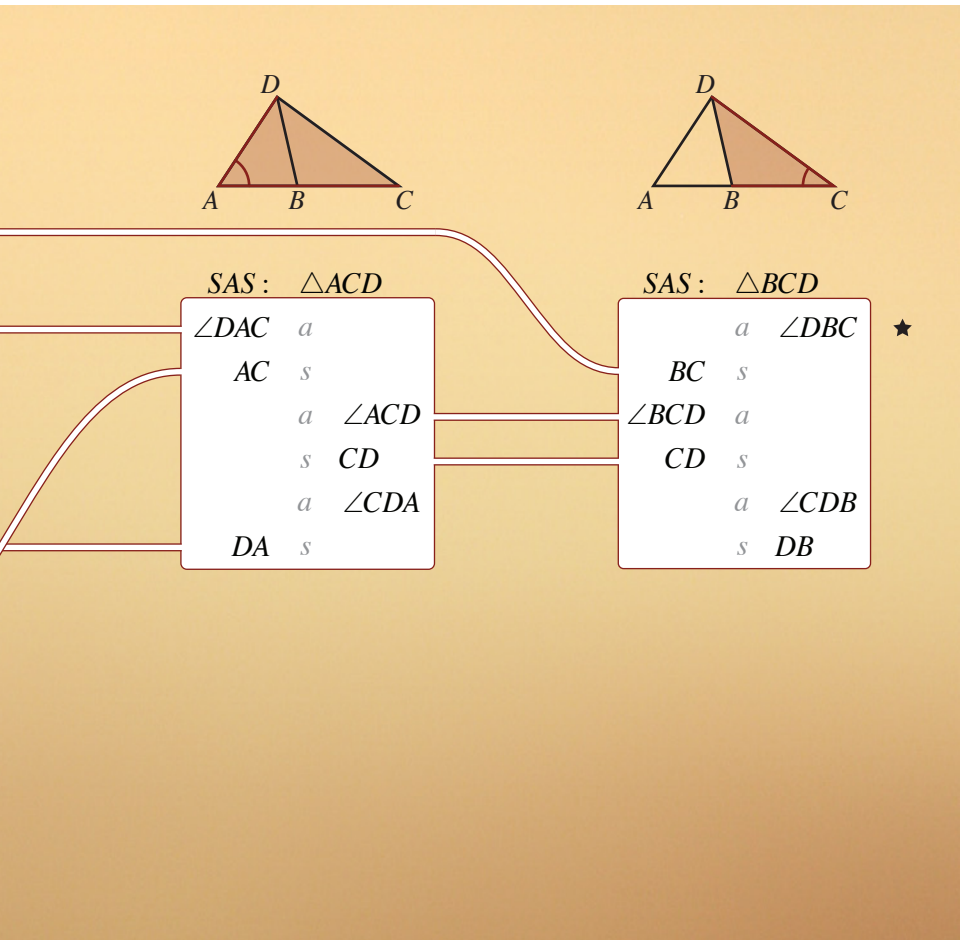if $\angle ABD \simeq \angle A'B'D'$, then $\angle CBD \simeq \angle C'B'D'$.

Proof. The idea is to relocate points to create a set of congruent triangles, and then to find a path of congruences leading from the given angles to the desired angle. In this case the relocation is easy enough: position A, C, and D on their respective rays $BA \rightarrow$, $BC \rightarrow$ and $BD \rightarrow$ so that

$$BA \simeq B'A' \quad BC \simeq B'C' \quad BD \simeq B'D'.$$

The path through the series of congruent triangles isn't that hard either if you just sit down to figure it out yourself. The problem is in writing it down so that a reader can follow along. In place of a traditional proof, I have made a chart that I think makes it easy to walk through the congruences. To read the chart, you need to know that I am using a little shorthand notation for each of the congruences. Here's the thing– each congruence throughout the entire proof compares segments, angles, or triangles with the *same* letters. The difference is that on the right hand side, the letters are marked with a $'$, while on the left they are not. For instance, the goal of this proof is to show that $\angle CBD \simeq \angle C'B'D'$. When I was working through the proof I found it a little tedious have to write the whole congruence out with every single step. Since the left hand side of
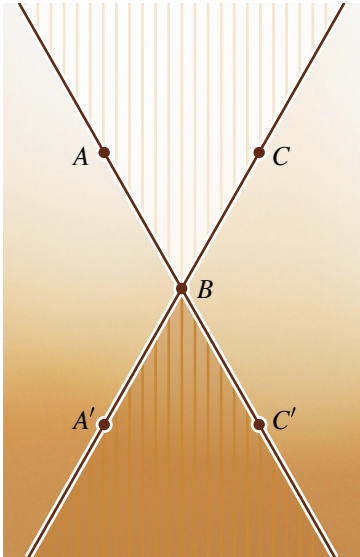
the congruence determines the right hand side anyway, I just got in the habit of writing down only the left hand side. In the end I decided that was actually easier to read than the whole congruence, so in the chart, the statement *AB* really means $AB \simeq A'B'$. I still feel a little uneasy doing this, so let me give another defense of this shorthand. One of the things I talked about in the last lesson was the idea of these congruences "locking in" a triangle– if you know S·A·S, for instance, then the triangle is completely determined. The statements in this proof can be interpreted as the locking in of various segments, angles, and triangles. For instance, *B* is between *A* and *C*, so if *AB* and *BC* are given, then *AC* is locked in by the Segment Addition Axiom. Okay, so that's enough about the notation. Here's the chart of the proof.

SAS :	△ACD		
∠DAC	a		
AC	s		
	a	∠ACD	
	s	CD	
	a	∠CDA	
DA	s		

SAS :	△BCD		
		a	∠DBC
BC	s		
∠BCD	a		
CD	s		
		a	∠CDB
	s	DB	

Every angle has two supplements. To get a supplement of an angle, simply replace one of the two rays forming the angle with its opposite ray. Since there are two candidates for this replacement, there are two supplements. There is a name for the relationship between these two supplements.

DEF: VERTICAL ANGLES
Vertical angles are two angles which are supplementary to the same angle.

Two intersecting lines generate two pairs of vertical angles.

Pair 1: ∠ABC and ∠A′BC′
Pair 2: ∠ABA′ and ∠CBC′

Every angle is part of one and only one vertical angle pair (something you may want to prove). For ∠ABC, the other half of the pair is the angle formed by the rays $(BA \to)^{op}$ and $(BC \to)^{op}$. Without a doubt, the single most important property of vertical angles is that

THM: ON VERTICAL ANGLES
Vertical angles are congruent.

Proof. Two vertical angles are, by definition, supplementary to the same angle. That angle is congruent to itself (because of the second axiom of congruence). Now we can use the last theorem. Since the vertical angles are supplementary to congruent angles, they themselves must be congruent.   □

The Alternate Interior Angle Theorem

The farther we go in the study of neutral geometry, the more we are going to bump into issues relating to how parallel lines behave. A lot of the results we will derive are maddeningly close to results of Euclidean geometry, and this can lead to several dangerous pitfalls. The Alternate Interior Angle Theorem is maybe the first glimpse of that.
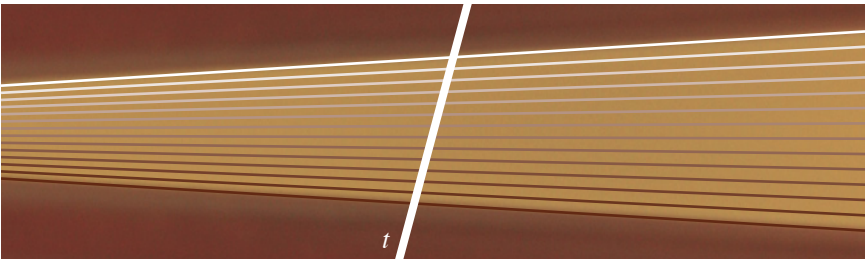
> DEF: TRANSVERSALS
> Given a set of lines, $\{\ell_1, \ell_2, \ldots, \ell_n\}$, a *transversal* is a line which intersects all of them.
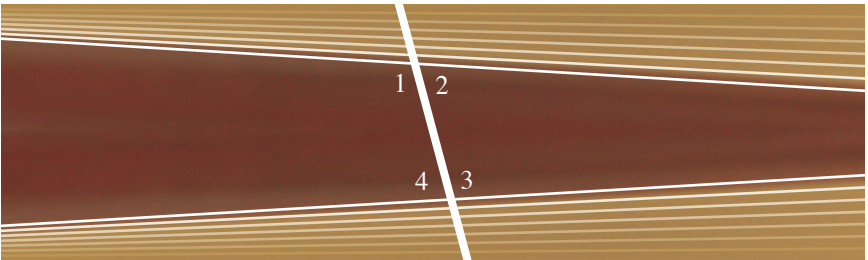
> DEF: ALTERNATE AND ADJACENT INTERIOR ANGLES
> Let t be a transversal to ℓ_1 and ℓ_2. *Alternate interior angles* are pairs of angles formed by ℓ_1, ℓ_2, and t, which are between ℓ_1 and ℓ_2, and on opposite sides of t. *Adjacent interior angles* are pairs of angles on the same side of t.

The Alternate Interior Angle Theorem tells us something about transversals and parallel lines. Read it carefully though. The converse of this theorem is used a lot in Euclidean geometry, but in neutral geometry this is *not* an "if and only if" statement.

A transversal t of a set of lines.

Alternate pairs: 1 *and* 3, 2 *and* 4. *Adjacent pairs:* 1 *and* 4, 2 *and* 3.

THE ALTERNATE INTERIOR ANGLE THEOREM
Let ℓ_1 and ℓ_2 be two lines, crossed by a transversal t.
If the alternate interior angles formed are congruent,
then ℓ_1 and ℓ_2 are parallel.

Proof. First I want to point out something that may not be entirely clear
in the statement of the theorem. The lines ℓ_1, ℓ_2 and t will actually form
two pairs of alternate interior angles. However, the angles in one pair are
the supplements of the angles in the other pair, so if the angles in one
pair are congruent then angles in the other pair also have to be congruent.
Now let's get on with the proof, a proof by contradiction. Suppose that
ℓ_1 and ℓ_2 are crossed by a transversal t so that alternate interior angles are
congruent, but suppose that ℓ_1 and ℓ_2 are not parallel. Label

A: the intersection of ℓ_1 and t;

B: the intersection of ℓ_2 and t;

C: the intersection of ℓ_1 and ℓ_2.

By the Segment Construction Axiom there are also points
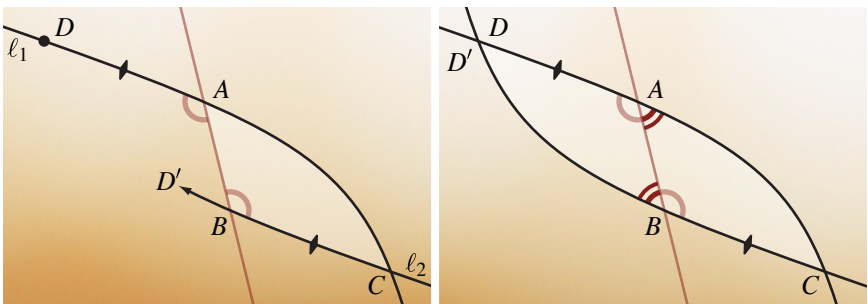
D on ℓ_1 so that $D * A * C$ and so that $AD \simeq BC$, and

D' on ℓ_2 so that $D' * B * C$ and so that $BD' \simeq AC$.

In terms of these marked points the congruent pairs of alternate interior
angles are
$$\angle ABC \simeq \angle BAD \quad \& \quad \angle ABD' \simeq \angle BAC.$$

Take the first of those congruences, together with the fact that that we
have constructed $AD \simeq BC$ and $AB \simeq BA$, and that's enough to use S·A·S:

If ℓ_1 and ℓ_2 crossed on one side of t, they would have to cross on the other side.
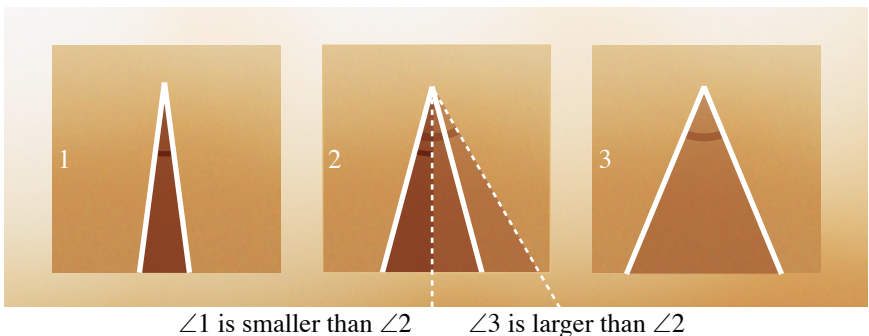
$\triangle ABC \simeq \triangle BAD$. I really just want to focus on one pair of corresponding angles in those triangles though: $\angle ABD \simeq \angle BAC$. Now $\angle BAC$ is congruent to its alternate interior pair $\angle ABD'$, so since angle congruence is transitive, this means that $\angle ABD \simeq \angle ABD'$. Here's the problem. There is only one way to construct this angle on that side of t, so the rays $BD{\rightarrow}$ and $BD'{\rightarrow}$ must actually be the same. That means that D, which we originally placed on ℓ_1, is also on ℓ_2. That would imply that ℓ_1 and ℓ_2 share two points, C and D, in violation of the very first axiom of incidence.   □

The Exterior Angle Theorem

We have talked about congruent angles, but so far we have not discussed any way of saying that one angle is larger or smaller than the other. That is something that we will need to do eventually, in order to develop a system of measurement for angles. For now though, we need at least some rudimentary definitions of this, even if the more fully developed system will wait until later.
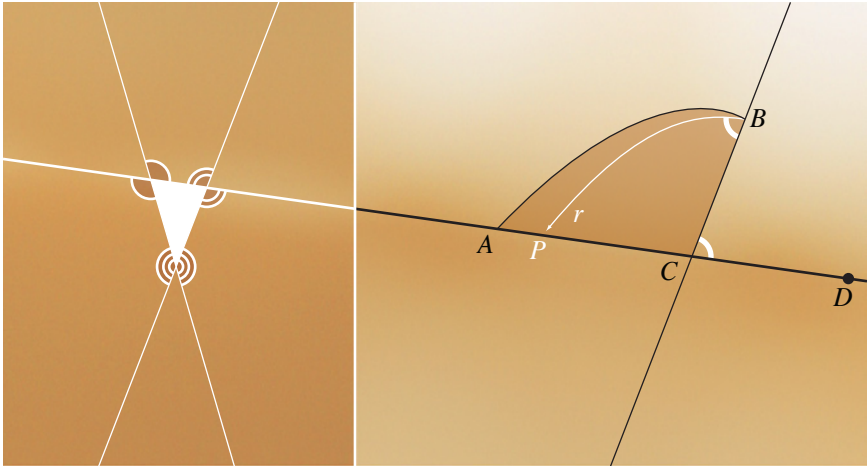
DEF: SMALLER AND LARGER ANGLES
Given two angles $\angle A_1B_1C_1$ and $\angle A_2B_2C_2$, the Angle Construction Axiom guarantees that there is a point $A^{\star}$ on the same side of ${\leftarrow}B_2C_2{\rightarrow}$ as A_2 so that $\angle A^{\star}B_2C_2 \simeq \angle A_1B_1C_1$. If $A^{\star}$ is in the interior of $\angle A_2B_2C_2$, then we say that $\angle A_1B_1C_1$ is *smaller* than $\angle A_2B_2C_2$. If $A^{\star}$ is on the ray B_2C_2, then the two angles are congruent as we have previously seen. If $A^{\star}$ is neither in the interior of $\angle A_2B_2C_2$, nor on the ray $B_2C_2{\rightarrow}$, then $\angle A_1B_1C_1$ is *larger* than $\angle A_2B_2C_2$.

∠1 is smaller than ∠2   ∠3 is larger than ∠2

In lesson 8, I will come back to this in more detail. Feel free to skip ahead if you would like a more detailed investigation of this way of comparing non-congruent angles.

DEF: EXTERIOR ANGLES
An *exterior angle* of a triangle is an angle supplementary to one of the triangle's interior angles.

Three pairs of exterior angles   Exterior Angle Th'm: a proof by contradiction

THE EXTERIOR ANGLE THEOREM
The measure of an exterior angle of a triangle is greater than the measure of either of the nonadjacent interior angles.

Proof. I will use a straightforward proof by contradiction. Starting with the triangle $\triangle ABC$, extend the side AC past C: just pick a point D so that $A * C * D$. Now suppose that the interior angle at B is larger than the exterior angle at $\angle BCD$. Then there is a ray r from B on the same side of BC as A so that $BC \rightarrow$ and r form an angle congruent to $\angle BCD$. This ray will lie in the interior of $\angle B$, though, so by the Crossbar Theorem, r must intersect AC. Call this intersection point P. Now wait, though. The alternate interior angles $\angle PBC$ and $\angle BCD$ are congruent. According to the Alternate Interior Angle Theorem r and AC must be parallel– they can't intersect. This is an contradiction.   □
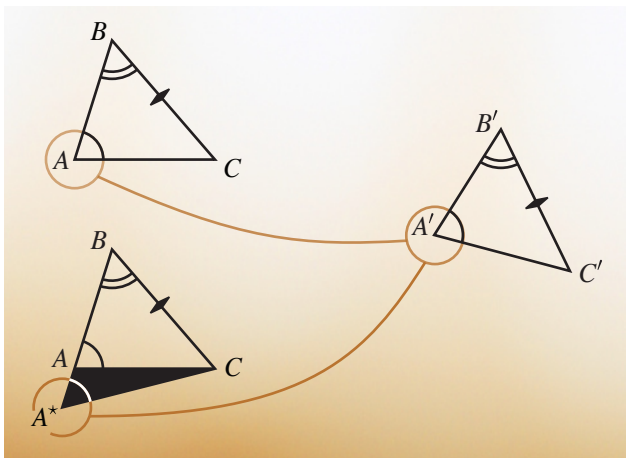
A·A·S TRIANGLE CONGRUENCE
In triangles $\triangle ABC$ and $\triangle A'B'C'$, if

$$\angle A \simeq \angle A' \quad \angle B \simeq \angle B' \quad BC \simeq B'C',$$

then $\triangle ABC \simeq \triangle A'B'C'$.

Proof. The setup of this proof is just like the proof of A·S·A, but for the critical step we are going to need to use the Exterior Angle Theorem. Locate $A^\star$ on $BA \rightarrow$ so that $A^\star B \simeq A'B'$. By S·A·S, $\triangle A^\star BC \simeq \triangle A'B'C'$. Therefore $\angle A^\star \simeq \angle A' \simeq \angle A$. Now if $B * A * A^\star$ (as illustrated) then $\angle A$ is an exterior angle and $\angle A^\star$ is a nonadjacent interior angle of the triangle $\triangle AA^\star C$. Acording to the Exterior Angle Theorem, these angles can't be congruent. If $B * A * A^\star$, then $\angle A^\star$ is an exterior angle and $\angle A$ is a nonadjacent interior angle. Again, the Exterior Angle Theorem says these angles can't be congruent. The only other possibility, then, is that $A = A^\star$, so $AB \simeq A'B'$, and by S·A·S, that means $\triangle ABC \simeq \triangle A'B'C'$. $\qquad \square$

A familiar chase to prove AAS, but this time we have to call upon the Exterior Angle Theorem.

Exercises

1. Prove that for every segment AB there is a point M on AB so that $AM \simeq MB$. This point is called the *midpoint* of AB.

2. Prove that for every angle $\angle ABC$ there is a ray $BD \rightarrow$ in the interior of $\angle ABC$ so that $\angle ABD \simeq \angle DBC$. This ray is called the *bisector* of $\angle ABC$.

3. Working from the spaghetti diagram proof that the supplements of congruent angles are congruent, write a traditional proof.