



**5. CONGRUENCE VERSE III**  
**OBJECTIVE: SSS**



In the last lesson I pointed out that the first and second axioms of congruence have angle counterparts in the fourth and fifth axioms, but that there was no direct angle counterpart to the third axiom, the Segment Addition Axiom. The next couple of results fill that hole.



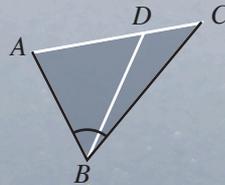
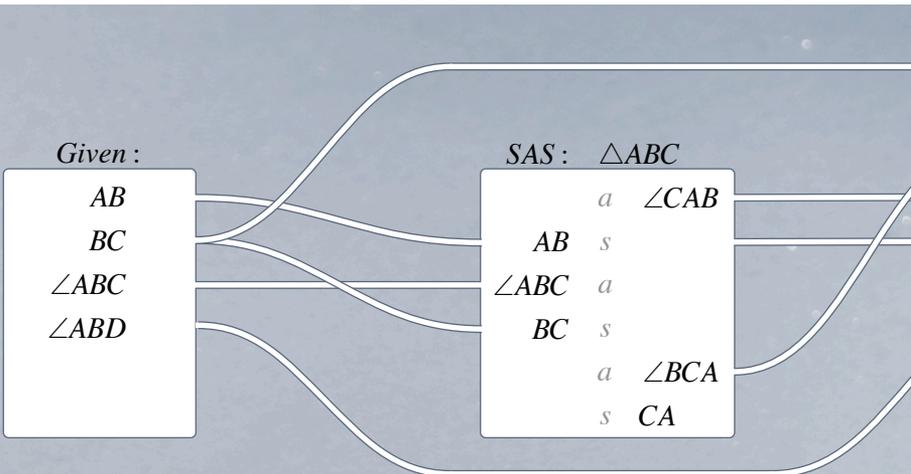
THE ANGLE SUBTRACTION THEOREM

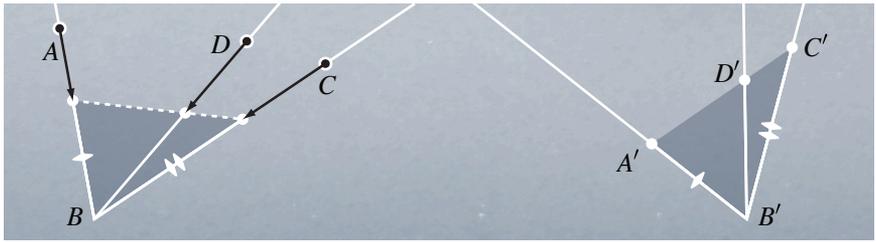
Let  $D$  and  $D'$  be interior points of  $\angle ABC$  and  $\angle A'B'C'$  respectively. If

$$\angle ABC \simeq \angle A'B'C' \quad \& \quad \angle ABD \simeq \angle A'B'D',$$

then  $\angle DBC \simeq \angle D'B'C'$ .

*Proof.* This proof is a lot like the proof that supplements of congruent angles are congruent, and I am going to take the same approach. The first step is one of relocation. Relocate  $A$  and  $C$  on  $BA \rightarrow$  and  $BC \rightarrow$  respectively





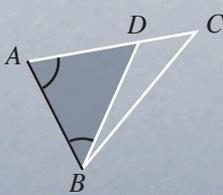
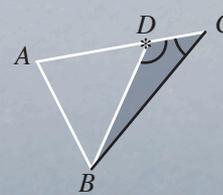
so that

$$BA \simeq B'A' \quad \& \quad BC \simeq B'C'.$$

Since  $D$  is in the interior of  $\angle ABC$ , by the Crossbar Theorem,  $BD \rightarrow$  intersects  $AC$ . Relocate  $D$  to that intersection. Likewise, relocate  $D'$  to the intersection of  $B'D' \rightarrow$  and  $A'C'$ . Note that this does not mean that  $BD \simeq B'D'$  although that is something that we will establish in the course of the proof. I am going to use a chart to illustrate the congruences in place of a “formal” proof.  $\square$

ASA: $\triangle ABD$	AAS: $\triangle BCD$
$\angle DAB$ <i>a</i>	<i>a</i> $\angle DBC$ ★
$AB$ <i>s</i>	$BC$ <i>s</i>
$\angle ABD$ <i>a</i>	$\angle BCD$ <i>a</i>
<i>s</i> $BD$	<i>s</i> $CD$
<i>a</i> $\angle BDA$	$\angle CDB$ <i>a</i>
<i>s</i> $DA$	<i>s</i> $DB$

\*if angles are congruent,  
their supplements are too.

With angle subtraction in the toolbox, angle addition is now easy to prove.



THE ANGLE ADDITION THEOREM

Suppose that  $D$  is in the interior of  $\angle ABC$  and that  $D'$  is in the interior of  $\angle A'B'C'$ . If

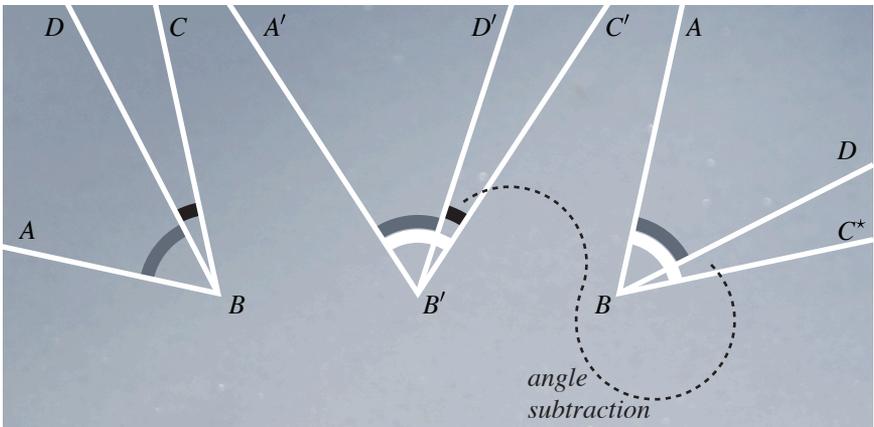
$$\angle ABD \simeq \angle A'B'D' \quad \& \quad \angle DBC \simeq \angle D'B'C',$$

then  $\angle ABC \simeq \angle A'B'C'$ .

*Proof.* Because of the Angle Construction Axiom, there is a ray  $BC^* \rightarrow$  on the same side of  $\leftarrow AB \rightarrow$  as  $C$  so that  $\angle ABC^* \simeq \angle A'B'C'$ . What we will show here is that  $BC \rightarrow$  and  $BC^* \rightarrow$  are actually the same so that the angles  $\angle ABC$  and  $\angle ABC^*$  are the same as well. This all boils down to one simple application of the Angle Subtraction Theorem:

$$\angle ABC^* \simeq \angle A'B'C' \quad \& \quad \angle ABD \simeq \angle A'B'D' \quad \implies \quad \angle DBC^* \simeq \angle D'B'C'.$$

We already know that  $\angle D'B'C' \simeq \angle DBC$ , so  $\angle DBC^* \simeq \angle DBC$ . The Angle Construction Axiom tells us that there is but one way to construct this angle on this side of  $\leftarrow DB \rightarrow$ , so  $BC^* \rightarrow$  and  $BC \rightarrow$  have to be the same.  $\square$



*The proof of the Angle Addition Theorem.*

We end this lesson with the last of the triangle congruence theorems. The proofs of the previous congruence theorems all used essentially the same approach, but that approach required an angle congruence. No angle congruence is given this time, so that won't work. Instead we are going to be using the Isosceles Triangle Theorem.



S · S · S TRIANGLE CONGRUENCE  
 In triangles  $\triangle ABC$  and  $\triangle A'B'C'$  if

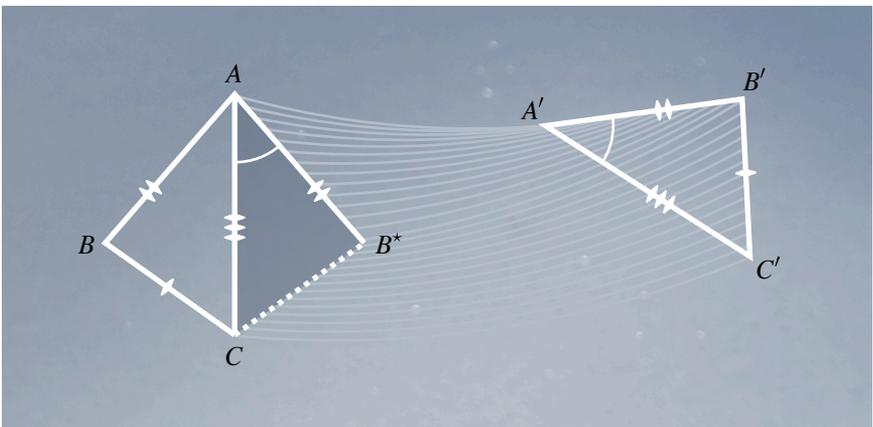
$$AB \simeq A'B' \quad BC \simeq B'C' \quad CA \simeq C'A',$$

then  $\triangle ABC \simeq \triangle A'B'C'$ .

*Proof.* The first step is to get the two triangles into a more convenient configuration. To do that, we are going to create a congruent copy of  $\triangle A'B'C'$  on the opposite side of  $\overleftrightarrow{AC}$  from  $B$ . The construction is simple enough: there is a unique point  $B^*$  on the opposite side of  $\overleftrightarrow{AC}$  from  $B$  such that:

$$\angle CAB^* \simeq \angle C'A'B' \quad \& \quad AB^* \simeq A'B'.$$

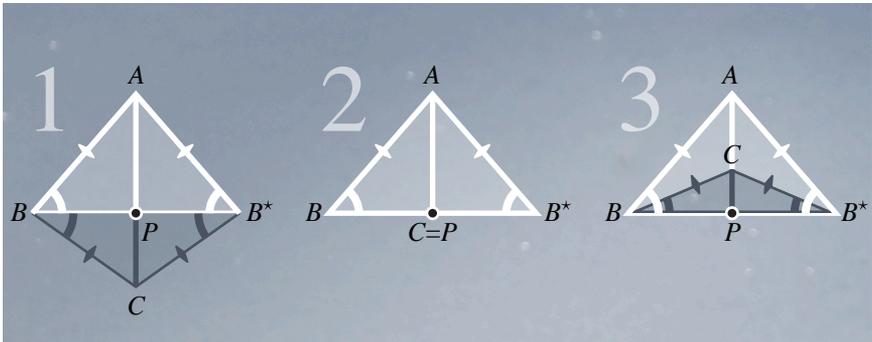
In addition, we already know that  $AC \simeq A'C'$ , so by S·A·S,  $\triangle ABC^*$  is congruent to  $\triangle A'B'C'$ . Now the real question is whether  $\triangle ABC^*$  is congruent to  $\triangle ABC$ , and that is the next task.



*Creating a congruent copy of the second triangle abutting the first triangle.*

Since  $B$  and  $B^*$  are on opposite sides of  $\leftarrow AC \rightarrow$ , the segment  $BB^*$  intersects  $\leftarrow AC \rightarrow$ . Let's call that point of intersection  $P$ . Now we don't know anything about where  $P$  is on  $\leftarrow AC \rightarrow$ , and that opens up some options:

- (1)  $P$  could be between  $A$  and  $C$ , or
- (2)  $P$  could be either of the endpoints  $A$  or  $C$ , or
- (3)  $P$  could be on the line  $\leftarrow AC \rightarrow$  but not the segment  $AC$ .



Three possible locations of  $P$ , and the resulting isosceles triangles.

I am just going to deal with that first possibility. If you want a complete proof, you are going to have to look into the remaining two cases yourself. Assuming that  $A * P * C$ , both of the triangles  $\triangle ABB^*$  and  $\triangle CBB^*$  are isosceles:

$$AB \simeq A'B' \simeq AB^*$$

$$CB \simeq C'B' \simeq CB^*.$$

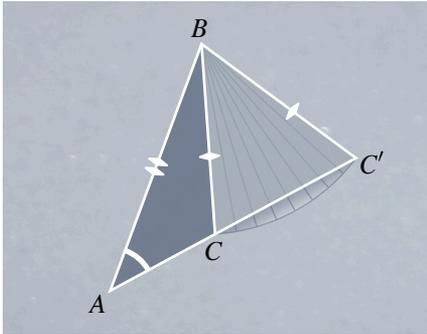
According to the Isosceles Triangle Theorem, the angles opposite those congruent sides are themselves congruent:

$$\angle ABP \simeq \angle AB^*P$$

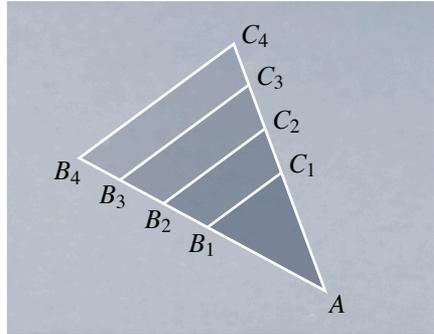
$$\angle CBP \simeq \angle CB^*P.$$

Since we are assuming that  $P$  is between  $A$  and  $C$ , we can use the Angle Addition Theorem to combine these two angles into the larger angle  $\angle ABC \simeq \angle AB^*C$ . We already know  $\angle AB^*C \simeq \angle A'B'C'$ , so  $\angle ABC \simeq \angle A'B'C'$  and that is the needed angle congruence. By S·A·S,  $\triangle ABC \simeq \triangle A'B'C'$ . □

*Failures of S·S·A and A·A·A in the Euclidean model.*



*Two noncongruent triangles sharing S·S·A.*



*Four noncongruent triangles sharing A·A·A.*

We have established four triangle congruences: S·A·S, A·S·A, A·A·S, and S·S·S. For each, you need three components, some mix of sides and angles. It would be natural to wonder whether there are any other combinations of three sides and angles which give a congruence. There are really only two other fundamentally different combinations: A·A·A and S·S·A. Neither is a valid congruence theorem in neutral geometry. In fact, both fail in Euclidean geometry. The situation in non-Euclidean geometry is a little bit different, but I am going to deflect that issue for the time being.

## Exercises

1. The Segment Addition Axiom. The Angle Subtraction Theorem. The Angle Addition Theorem. That just leaves the Segment Subtraction Theorem. State it. Prove it.
2. One of the conditions in the statement of the Angle Subtraction Theorem is that both  $D$  and  $D'$  must be in the interiors of their respective angles. In fact, this condition can be weakened: prove that you do not need to assume that  $D'$  is in the interior of the angle, just that it is on the same side of  $A'B'$  as  $C'$ .
3. Complete the proof of S·S·S by handling the other two cases (when  $P$  is one of the endpoints and when  $P$  is on the line  $\leftarrow AC \rightarrow$  but not the segment  $AC$ ).
4. Suppose that  $A*B*C$  and that  $A'$  and  $C'$  are on opposite sides of  $\leftarrow AC \rightarrow$ . Prove that if  $\angle ABA' \simeq \angle CBC'$ , then  $A'*B*C'$ .
5. Suppose that  $A, B, C$ , and  $D$  are four distinct non-collinear points. Prove that if  $\triangle ABC \simeq \triangle DCB$ , then  $\triangle BAD \simeq \triangle CDA$ .



