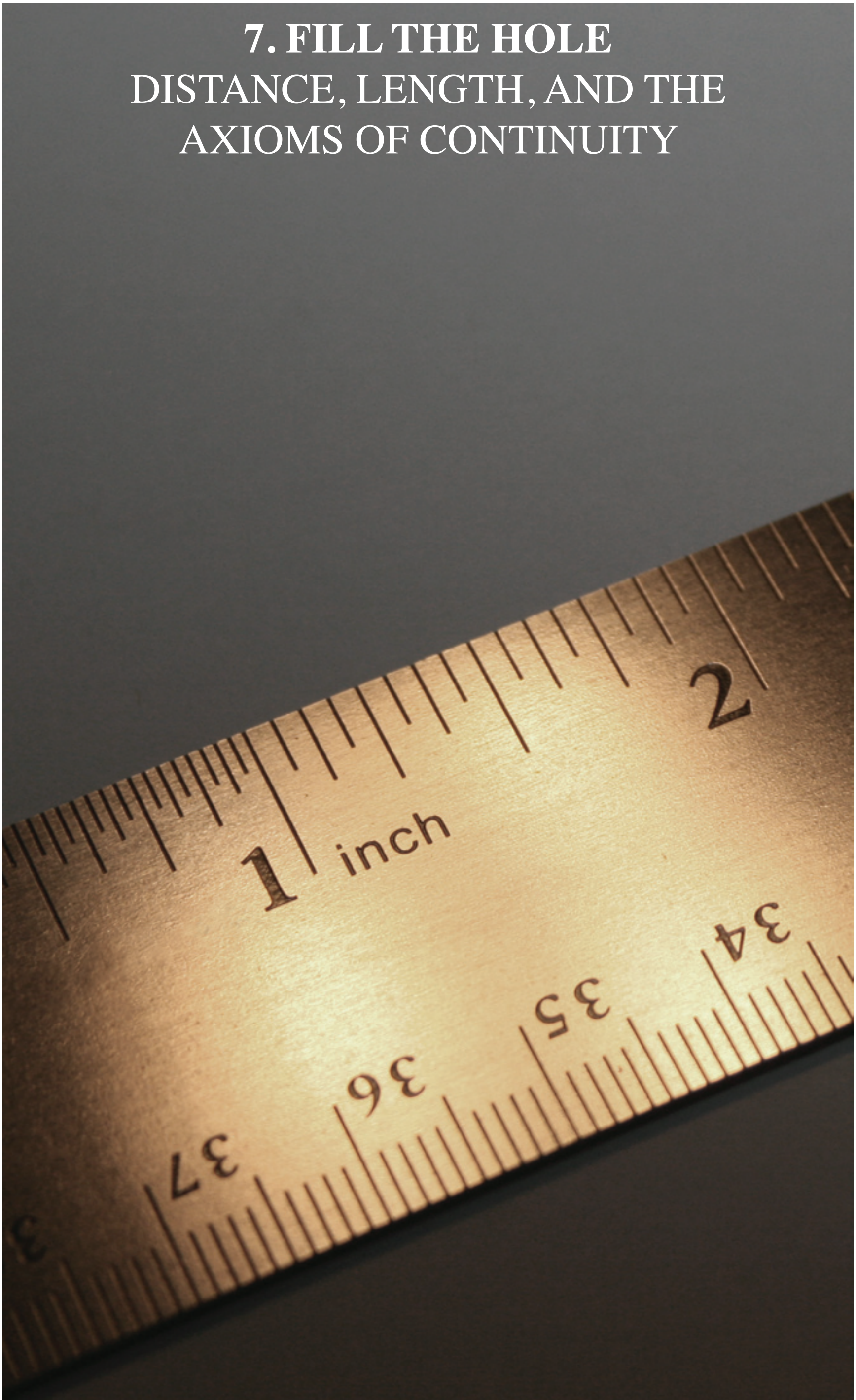
7. FILL THE HOLE
DISTANCE, LENGTH, AND THE AXIOMS OF CONTINUITY



Hilbert's geometry starts with incidence, congruence, and order. It is a synthetic geometry in the sense that it is not centrally built upon measurement. Nowadays, it is more common to take an metrical approach to geometry, and to establish your geometry based upon a measurement. In the metrical approach, you begin by defining a distance function– a function d which assigns to each pairs of points a real number and satisfies the following requirements

(i) $d(P,Q) \geq 0$, with $d(P,Q) = 0$ if and only if $P = Q$,
(ii) $d(P,Q) = d(Q,P)$, and
(iii) $d(P,R) \leq d(P,Q) + d(Q,R)$.

Once the distance function has been chosen, the length of a segment is defined to be the distance between its endpoints. I will follow the convention of using the absolute value sign to notate the length of a segment, so $|PQ| = d(P,Q)$. Then congruence is defined by saying that two segments are congruent if they have the same length. Incidence and order also can be defined in terms of d: points P, Q, and R are all on the same line, and Q is between P and R when the inequality in (iii) is an equality. You see, synthetic geometry takes a back seat to analytic geometry, and the synthetic notions of incidence, order, and congruence, are defined analytically. I do not have a problem with that approach– it is the one that we are going to take in the development of hyperbolic geometry much later on. We have been developing a synthetic geometry, though, and so what I would like to do in this lesson is to build distance out of incidence, order, and congruence. This is what Hilbert did when he developed the real number line and its properties inside of the framework of his axiomatic system.

Modest Expectations

Here we stand with incidence, order, congruence, the axioms describing them, and at this point even a few theorems. Before we get out of this section, I will throw in the last two axioms of neutral geometry, the axioms of continuity, too. From all of this, we want to build a distance function d. Look, we have all dealt with distance before in one way or another, and we want our distance function to meet conditions (i)–(iii) above, so it is fair to have certain expectations for d. I don't think it is unreasonable to expect all of the following.

(1) The distance between any two distinct points should be a positive real number and the distance from a point to itself should be zero. That way, d will satisfy condition (i) above.

(2) Congruent segments should have the same length. That takes care of condition (ii) above, since $AB \simeq BA$, but it does a whole lot more too. You see, let's pick out some ray r and label its endpoint O. According to the Segment Construction Axiom, for any segment AB, there is a unique point P on r so that $AB \simeq OP$. If congruent segments are to have the same length, then that means $|AB| = d(O,P)$. Therefore, if we can just work out the distance from O to the other points on r, then all other distances will follow.

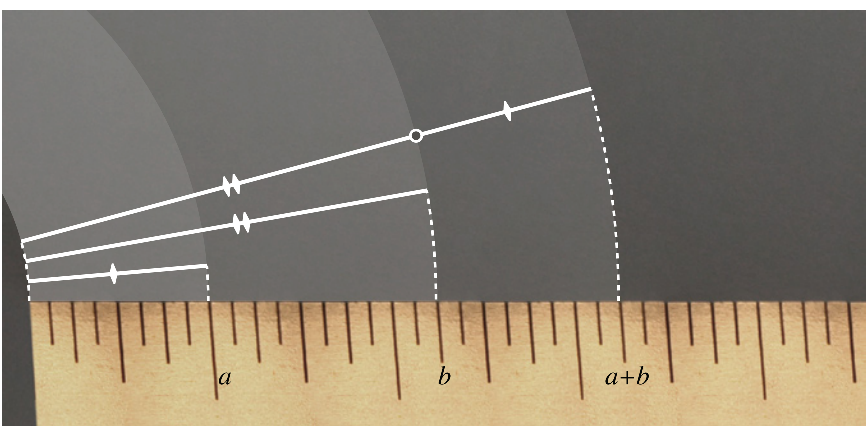
(3) If $A * B * C$, then

$$|AB| + |BC| = |AC|.$$

This is just a part of property (iii) of a distance function. Since we are going to develop the distance function on r, we don't have to worry about non-colinear points just yet (that will come a little later). Relating back to your work in the last section, since d never assigns negative values, this means that

$$\begin{aligned} AB \prec CD &\implies |AB| < |CD|, \\ AB \succ CD &\implies |AB| > |CD|. \end{aligned}$$

It is up to us to build a distance function that meets all three of these requirements. The rest of this chapter is devoted to doing just that.



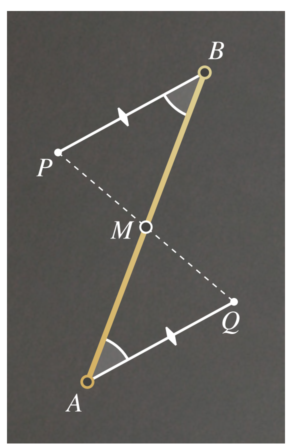
The additivity condition for d.

Divide and combine: the dyadic points

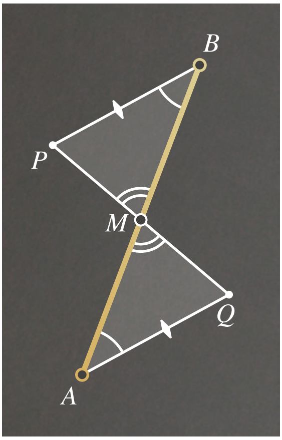
With those conditions in mind, let's start building the distance function d. The picture that I like to keep in my mind as I'm doing this is that simple distance measuring device: the good old-fashioned ruler. Not a metric ruler mind you, but an English ruler with inches on it. Here is one way that you can classify the markings on the ruler. You have the $1''$ mark. That distance is halved, and halved, and halved again to get the $1/2''$, $1/4''$, and $1/8''$ marks. Depending upon the precision of the ruler, there may be $1/16''$ or $1/32''$ markings as well. All the other marks on the ruler are multiples of these. Well, that ruler is the blueprint for how we are going to build the skeleton of d. First of all, because of condition (1), $d(O,O) = 0$. Now take a step along r to another point. Any point is fine– like the inch mark on the ruler, it sets the unit of measurement. Call this point P_0 and define $d(O,P_0) = 1$. Now, as with the ruler, we want to repeatedly halve OP_0. That requires a little theory.

DEF: MIDPOINT
A point M on a segment AB is the *midpoint* of AB if $AM \simeq MB$.

THM: EXISTENCE, UNIQUENESS OF MIDPOINTS
Every segment has a unique midpoint.



Proof. *Existence*. Given the segment AB, choose a point P which is *not* on $\leftarrow AB \rightarrow$. According to the Angle and Segment Construction Axioms, there is a point Q on the opposite side of $\leftarrow AB \rightarrow$ from P so that $\angle ABP \simeq \angle BAQ$ (that's the angle construction part) and so that $BP \simeq AQ$ (that's the segment construction part). Since P and Q are on opposite sides of $\leftarrow AB \rightarrow$, the segment PQ intersects it. Call that point of intersection M. I claim that M is the midpoint of AB. Why? Well, compare $\triangle MBP$ and $\triangle MAQ$.



In those triangles

$$\begin{aligned} \angle AMQ &\simeq \angle BMP && \text{(vertical angles)} \\ \angle MAQ &\simeq \angle MBP && \text{(by construction)} \\ BP &\simeq AQ && \text{(by construction)} \end{aligned}$$

so, by A·A·S, they must be congruent triangles. That means that $AM \simeq MB$. It is worth noting that the midpoint of AB has to be between A and B. If it weren't, one of two things would have to happen:

$$\begin{aligned} M * A * B &\implies MA \prec MB, \text{ or} \\ A * B * M &\implies MA \succ MB, \end{aligned}$$

and either way, the segments MA and MB couldn't be congruent.

Uniqueness. Suppose that a segment AB actually had two midpoints. Let's call them M_1 and M_2, and just for the sake of convenience, let's say that they are labeled so that they are ordered as

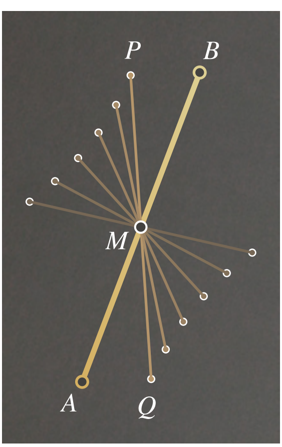
$$A * M_1 * M_2 * B.$$

Since $A * M_1 * M_2$, $AM_1 \prec AM_2$. Since $M_1 * M_2 * B$, $BM_2 \prec BM_1$. But now M_2 is a midpoint, so $AM_2 \simeq BM_2$. Let's put that together

$$AM_1 \prec AM_2 \simeq BM_2 \prec BM_1.$$

In the last section you proved that $\prec$ is transitive. This would imply that $AM_1 \prec BM_1$ which contradicts the fact that M_1 is a midpoint. Hence a segment cannot have two distinct midpoints. $\square$

There are many choices for P, but they each lead to the same midpoint because a segment can have only one midpoint.



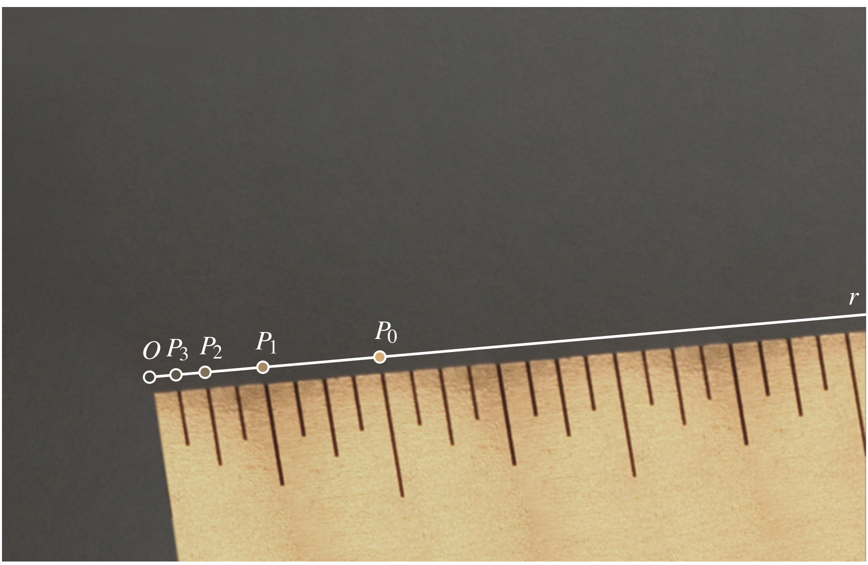
Let's go back to OP_0. We now know that it has a unique midpoint. Let's call that point P_1. In order for the distance function d to satisfy condition (3),

$$|OP_1| + |P_1P_0| = |OP_0|.$$

But OP_1 and P_1P_0 are congruent, so in order for d to satisfy condition (2), they have to be the same length. Therefore $2|OP_1| = 1$ and so $|OP_1| = 1/2$. Repeat. Take OP_1, and find its midpoint. Call it P_2. Then

$$|OP_2| + |P_2P_1| = |OP_1|.$$

Again, OP_2 and P_2P_1 are congruent, so the must be the same length. Therefore $2|OP_2| = 1/2$, and so $|OP_2| = 1/4$. By repeating this process over and over, you can identify the points P_n which are distances of $1/2^n$ from O.



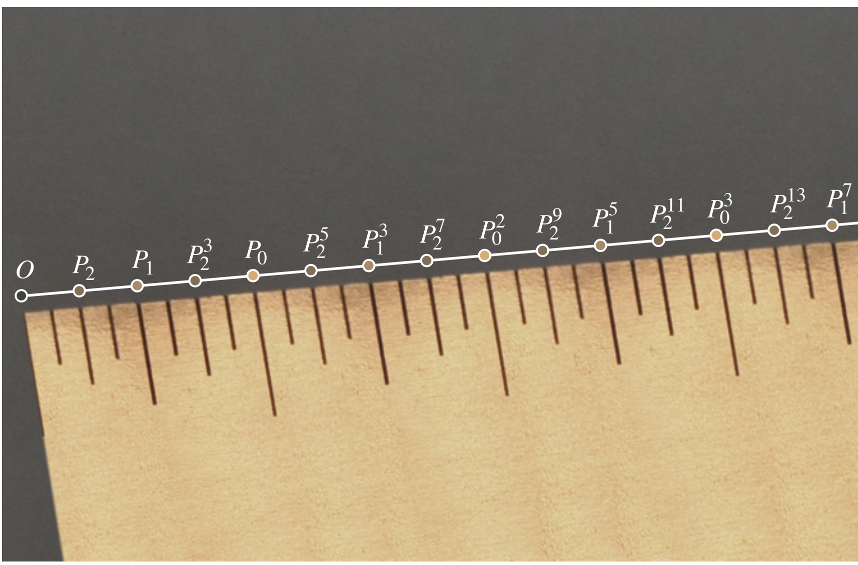
With the points P_n as building blocks, we can start combining segments of lengths $1/2^n$ to get to other points. In fact, we can find a point whose distance from O is $m/2^n$ for any positive integers m and n. It is just a matter of chaining together enough congruent copies of OP_n as follows. Begin with the point P_n. By the first axiom of congruence, there is a point P_n^2 on the opposite side of P_n from O so that $P_nP_n^2 \simeq OP_n$. And there is a point P_n^3 on the opposite side of P_n^2 from P_n so that $P_n^2P_n^3 \simeq OP_n$. And a point P_n^4 on the opposite side of P_n^3 from P_n^2 so that $P_n^3P_n^4 \simeq OP_n$. And so on. This can be continued until m segments are chained together stretching from O to a point which we will label P_n^m. In order for the distance function to satisfy the additivity condition (3),

$$|OP_n^m| = |OP_n| + |P_nP_n^2| + |P_n^2P_n^3| + \cdots + |P_n^{m-1}P_n^m|.$$

All of these segments are congruent, though, so they have to be the same length (for condition (2)), so

$$|OP_n^m| = m \cdot |OP_n| = m \cdot 1/2^n = m/2^n.$$

Rational numbers whose denominator can be written as a power of two are called dyadic rationals. In that spirit, I will call these points the dyadic points of r.



Fill the Hole

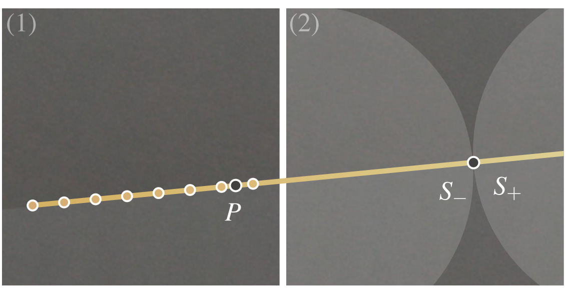
There are plenty of real numbers that aren't dyadic rationals though, and there are plenty of points on r that aren't dyadic points. How can we measure the distance from O to them? For starters, we are not going to be able to do this without the last two axioms of neutral geometry.

These last two axioms, the axioms of continuity, are a little more technical than any of the previous ones. The first says that you can get to any point on a line if you take enough steps. The second, which is inspired by Dedekind's construction of the real numbers, says that there are no gaps in a line.

THE AXIOMS OF CONTINUITY

Ct1 *Archimedes' Axiom* If AB and CD are any two segments, there is some positive integer n such that n congruent copies of CD constructed end-to-end from A along the ray $AB\rightarrow$ will pass beyond B.

Ct2 *Dedekind's Axiom* Let $\mathbb{S}_<$ and $\mathbb{S}_\geq$ be two nonempty subsets of a line ℓ satisfying: (i) $\mathbb{S}_< \cup \mathbb{S}_\geq = \ell$; (ii) no point of $\mathbb{S}_<$ is between two points of $\mathbb{S}_\geq$; and (iii) no point of $\mathbb{S}_\geq$ is between two points of $\mathbb{S}_<$. Then there is a unique point O on ℓ such that for any two other points P_1 and P_2 with $P_1 \in \mathbb{S}_<$ and $P_2 \in \mathbb{S}_\geq$ then $P_1 * O * P_2$



(1) *Archimedes: Given enough steps, P will be passed.*

(2) *Dedekind: There is a point between any two separated partitions of a line.*

It is time to get back to the issue of distance on the ray r. So let P be a point on r. Even if P is not itself a dyadic point, it is surrounded by dyadic points. In fact, there are so many dyadic points crowding P, that the distance from O to P can be estimated to any level of precision using nearby dyadic points. For instance, suppose we consider just the dyadic points whose denominator can be written as 2^0:

$$S_0 = \{O, P_0^1, P_0^2, P_0^3, \ldots\}.$$

By the Archimedean Axiom, eventually these points will lie beyond P. If we focus our attention on the one right before P, say $P_0^{m_0}$, and the one right after, $P_0^{m_0+1}$, then

$$O * P_0^{m_0} * P * P_0^{m_0+1}.$$

We can compare the relative sizes of the segments

$$OP_0^{m_0} \prec OP \prec OP_0^{m_0+1}$$

and so, if our distance is going to satisfy condition (3),

$$\begin{aligned} |OP_0^{m_0}| <& |OP| < |OP_0^{m_0+1}| \\ m_0 <& |OP| < m_0+1 \end{aligned}$$

Not precise enough for you? Replace S_0, with S_1, the set of dyadic points whose denominator can be written as 2^1:

$$S_1 = \{O, P_1, P_1^2 = P_0, P_1^3, P_1^4 = P_0^2, \ldots\}.$$

Again, the Archimedean Axiom guarantees that eventually the points in S_1 will pass beyond P. Let $P_1^{m_1}$ be the last one before that happens. Then

$$O * P_1^{m_1} * P * P_1^{m_1+1}$$

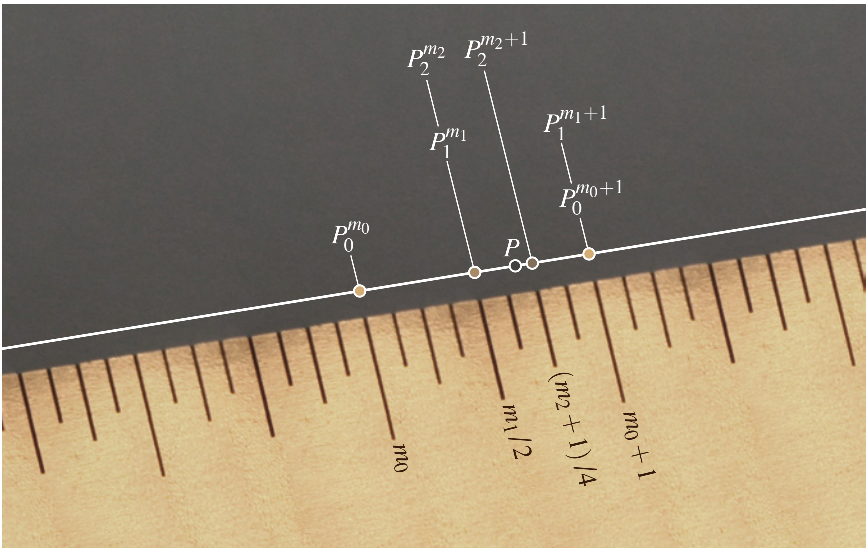
so

$$\begin{aligned} |OP_1^{m_1}| <& |OP| < |OP_1^{m_1+1}| \\ m_1/2 <& |OP| < (m_1+1)/2 \end{aligned}$$

and this gives $|OP|$ to within an accuracy of 1/2.

Continuing along in this way, you can use S_2, dyadics whose denominator can be written as 2^2, to approximate $|OP|$ to within $1/4$, and you can use S_3, dyadics whose denominator can be written as 2^3, to approximate $|OP|$ to within $1/8$. Generally speaking, the dyadic rationals in S_n provide an upper and lower bound for $|OP|$ which differ by $1/2^n$. As n goes to infinity, $1/2^n$ goes to zero, forcing the upper and lower bounds to come together at a single number. This number is going to have to be $|OP|$. Now you don't really need both the increasing and decreasing sequences of approximations to define $|OP|$. After all, they both end up at the same number. Here is the description of $|OP|$ using just the increasing sequence: for each positive integer n, let $P_n^{m_n}$ be the last point in the list S_n which is between O and P. In order for the distance function to satisfy condition (3), we must set

$$|OP| = \lim_{n\to\infty} |OP_n^{m_n}| = \lim_{n\to\infty} m_n/2^n.$$



Capturing a non-dyadic point between two sequences of dyadic points.

Now do it in reverse

Every point of r now has a distance associated with it, but is there a point at every possible distance? Do we know, for instance, that there is a point at exactly a distance of 1/3 from O? The answer is yes– it is just a matter of reversing the distance calculation process we just described and using the Dedekind Axiom. Let's take as our prospective distance some positive real number x. For each integer $n \geq 0$, let $m_n/2^n$ be the largest dyadic rational less than x whose denominator can be written as 2^n and let $P_n^{m_n}$ be the corresponding dyadic point on r. Now we are going to define two sets of points:

$\mathbb{S}_<$: all the points of r that lie between O and any of the $P_n^{m_n}$, together with all the points of r^{op}.

$\mathbb{S}_\geq$: all of the remaining points of r.

So $\mathbb{S}_<$ contains a sequence of dyadic rationals increasing to x

$$\{P_0^{m_0}, P_1^{m_1}, P_2^{m_2}, P_3^{m_3}, \ldots\},$$

and $\mathbb{S}_\geq$ contains a sequence of dyadic rationals decreasing to x

$$\{P_0^{m_0+1}, P_1^{m_1+1}, P_2^{m_2+1}, P_3^{m_3+1}, \ldots\}.$$

Together $\mathbb{S}_<$ and $\mathbb{S}_\geq$ contain all the points of the line through r, but they do not intermingle: no point of $\mathbb{S}_<$ is between two of $\mathbb{S}_\geq$ and no point of $\mathbb{S}_\geq$ is between two of $\mathbb{S}_<$. According to the Dedekind Axiom, then, there is a unique point P between $\mathbb{S}_<$ and $\mathbb{S}_\geq$. Now let's take a look at how far P is from O. For all n,

$$\begin{aligned} OP_n^{m_n} &\prec OP \prec OP_n^{m_n+1} \\ |OP_n^{m_n}| &< |OP| < |OP_n^{m_n+1}| \\ m_n/2^n &< |OP| < (m_n+1)/2^n \end{aligned}$$

As n goes to infinity, the interval between these two consecutive dyadics shrinks – ultimately, the only point left is x. So $|OP| = x$.

Example: dyadics approaching 1/3

Finding a dyadic sequence approaching a particular number can be tricky business. Finding such a sequence approaching 1/3 is easy, though, as long as you remember the geometric series formula

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{if } |x| < 1.$$

With a little trial and error, I found that by plugging in $x = 1/4$,

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{4}{3}.$$

Subtracting one from both sides gives an infinite sum of dyadics to 1/3, and we can extract the sequence from that

$$\begin{aligned} &\frac{1}{4} = 0.25 \\ &\frac{1}{4} + \frac{1}{16} = \frac{5}{16} = 0.3125 \\ &\frac{1}{4} + \frac{1}{16} + \frac{1}{64} = \frac{21}{64} = 0.32825 \end{aligned}$$

Segment addition, redux

For any two points P and Q, there is a unique segment OR on the ray r which is congruent to PQ. Define $d(P,Q) = |OR|$. With this setup, our distance function will satisfy conditions (1) and (2). That leaves condition (3)– a lot of effort went into trying to build d so that condition would be satisfied, but it is probably a good idea to make sure that it actually worked. Let's close out this lesson with two theorems that do that.

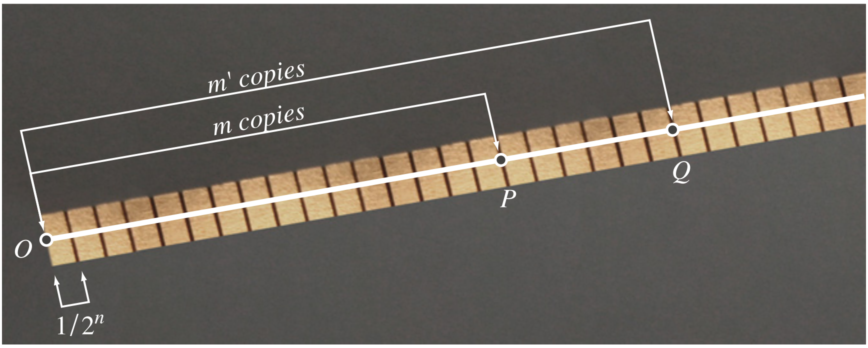
THM: A FORMULA FOR DISTANCE ALONG A RAY
If P and Q are points on r, with $|OP| = x$ and $|OQ| = y$, and if P is between O and Q, then $|PQ| = y - x$.

Proof. If both P and Q are dyadic points, then this is fairly easy. First you are going to express their dyadic distances with a common denominator:

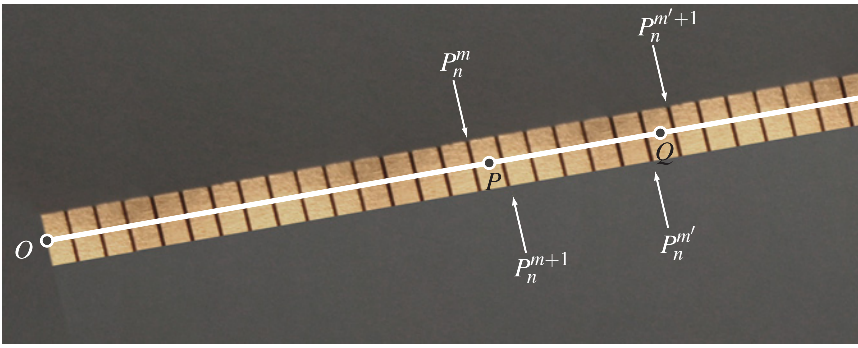
$$|OP| = m/2^n \quad |OQ| = m'/2^n.$$

Then OP is built from m segments of length $1/2^n$ and OQ is built from m' segments of length $1/2^n$. To get $|PQ|$, you simply have to take the m segments from the m' segments– so $|PQ|$ is made up of $m' - m$ segments of length $1/2^n$. That is

$$|PQ| = (m' - m) \cdot \frac{1}{2^n} = y - x.$$



Measuring the distance between two dyadic points.



Measuring the distance between two non-dyadic points.

If one or both of P and Q are not dyadic, then there is a bit more work to do. In this case, P and Q are approximated by a sequence of dyadics $P_n^{m_n}$ and $P_n^{m'_n}$ where

$$\lim_{n\to\infty}\frac{m_n}{2^n}=x \quad \& \quad \lim_{n\to\infty}\frac{m'_n}{2^n}=y.$$

Now we can trap $|PQ|$ between dyadic lengths:

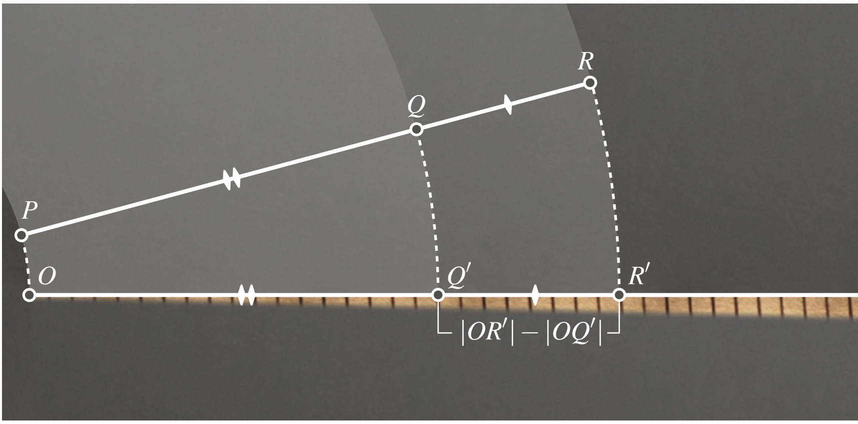
$$P_n^{m_n+1}P_n^{m'_n} \prec PQ \prec P_n^{m_n}P_n^{m'_n+1}$$
$$|P_n^{m_n+1}P_n^{m'_n}| < |PQ| < |P_n^{m_n}P_n^{m'_n+1}|$$
$$\frac{m'_n-m_n-1}{2^n} < |PQ| < \frac{m'_n+1-m_n}{2^n}$$

As n approaches infinity, $|PQ|$ is stuck between two values both of which are approaching $y-x$. □

Now while this result only gives a formula for lengths of segments on the ray r, it is easy to extend it to a formula for lengths of segments on the line containing r. In fact, this is one of the exercises for this lesson. The last result of this lesson is a reinterpretation of the Segment Addition Axiom in terms of distance, and it confirms that the distance we have constructed does satisfy condition (3).

THM: SEGMENT ADDITION, THE MEASURED VERSION
If $P*Q*R$, then $|PQ|+|QR|=|PR|$.

Proof. The first step is to transfer the problem over to r so that we can start measuring stuff. So locate Q' and R' on r so that:



$$O * Q' * R', \quad PQ \simeq OQ', \quad QR \simeq Q'R'.$$

According to the Segment Addition Axiom, this means that $PR \simeq OR'$. Now we can use the last theorem,

$$|QR| = |Q'R'| = |OR'| - |OQ'| = |PR| - |PQ|.$$

Just solve that for $|PR|$ and you've got it. □

Exercises

1. Our method of measuring distance along a ray r can be extended to the rest of the line. In our construction each point on r corresponds to a positive real number (the distance from O to that point). Suppose that P is a point on r^{op}. There is a point Q on r so that $OP \simeq OQ$. If x is the positive real number associated with Q, then we want to assign the negative number $-x$ to P. Now suppose that P_1 and P_2 are any two points on the line and x and y are the associated real numbers. Show that
$$d(P_1, P_2) = |x - y|.$$

2. Write $1/7$, $1/6$, and $1/5$ as an infinite sum of dyadic rationals.

3. Since writing this, it has come to my attention (via Greenberg's book [1]) that Archimedes' Axiom is actually a consequece of Dedekind's Axiom. You can prove this yourself as follows. If Archimedes were not true, then there would be some point on a ray that could not be reached by via end-to-end copies of a segment. In that case, the ray can be divided into two sets: one consisting of the points that can be reached, the other of the points that cannot. By including the opposite ray in with the set of points that can be reached, you get a partition of a line into two sets. Prove that these sets form a Dedekind cut of the line. Then by Dedekind's Axiom there is a point between them. Now consider what would happen if you took one step forward or backward from this point.

References

[1] Marvin J. Greenberg. *Euclidean and Non-Euclidean Geometries: Development and History*. W.H. Freeman and Company, New York, 4th edition, 2008.