9. READER'S SOLO II
ANGLE MEASURE

In this lesson I am going to outline what you need to do to construct the degree measurement system for angles. First, let's talk notation. I think the most common way to indicate the measure of an angle $\angle ABC$ is to write $m(\angle ABC)$. The advantage of that notation is that it draws a clear distinction between an angle and its measure. Of course, the disadvantage is that it is cumbersome, and that any equation with lots of angles measures in it will be cluttered up with m's. At the other extreme, I have noticed that students tend to just write the angle $\angle ABC$ to indicate its measure. Sure, it is just laziness, but I suppose you could pass it off as notational efficiency as well. The obvious disadvantage of this approach is that it completely blurs the distinction between an angle and its measure. I have tried to find the middle ground between these two approaches and I write $(\angle ABC)$ to denote the measure of $\angle ABC$. This notation is not perfect either. I think the biggest problem is that it puts even more pressure on two of the most overused symbols in mathematics, the parentheses.

Now lets talk about what you are going to want in a system of angle measurement. Of course these expectations are going to closely mirror expectations for measures of distance. They are

(1) The measure of an angle should be a positive real number.

(2) Congruent angles should have the same measure. That allows us to focus our investigation on just the angles which are built off of one fixed ray.

(3) If D is in the interior of $\angle ABC$, then

$$(\angle ABC) = (\angle ABD) + (\angle DBC).$$

Therefore, since the measure of an angle has to be positive,

$$\begin{aligned} \angle ABC \prec \angle A'B'C' &\implies (\angle ABC) < (\angle A'B'C') \\ \angle ABC \succ \angle A'B'C' &\implies (\angle ABC) > (\angle A'B'C'). \end{aligned}$$

It is your turn to develop a system of angle measure that will meet those requirements. The first step is to establish the measurement of dyadic angles. To do that, you will have to prove that it is possible to divide an angle in half.

DEF: ANGLE BISECTOR
For any angle $\angle ABC$, there is a unique ray $BD \rightarrow$ in the interior of $\angle ABC$ so that $\angle ABD \simeq \angle DBC$. This ray is called the *angle bisector* of $\angle ABC$.

With segment length, everything begins with an arbitrary segment which is assigned a length of one. With angle measure, everything begins with a right angle which, in the degree measurement system, is assigned a measure of 90°. From that, your next step is to describe the process of constructing angles with measures $90^\circ \cdot m/2^n$. Here you are going to run into one fundamental difference between angles and segments– segments can be extended arbitrarily, but angles cannot be put together to exceed a straight angle. Therefore segments can be arbitrarily long, but all angles must measure less than 180° (since a straight angle is made up of two right angles). It is true that the unit circle in trigonometry shows how you can loop back around to define angles with any real measure, positive or negative, and that is a useful extension in some contexts, but it also creates some problems (the measure of an angle is not uniquely defined, for instance).

Once you have figured out the dyadic angles, you need to fill in the rest. You will want to use a limiting process just like I did in the segment length chapter: this time the key word "interior" will replace the key word "between." Then you will want to turn the question around: for any real number in the interval $(0^\circ, 180^\circ)$ is there an angle with that as its measure? This is where I used the Dedekind Axiom before, by taking a limit of approximating dyadics, and then using the axiom to say that there is a point at that limit. The problem for you is that the Dedekind Axiom applies only to points on a line– it is not about angles (or at least not directly). Nevertheless, you need to find a way to set up approximating dyadic angles, and then you need to find some way to make Dedekind's Axiom apply in this situation.

Finally, with angles measured in this way, you will need to verify the additivity of angle measure:

THM: ANGLE ADDITION, THE MEASURED VERSION
If D is in the interior of $\angle ABC$, then $(\angle ABC) = (\angle ABD) + (\angle DBC)$.