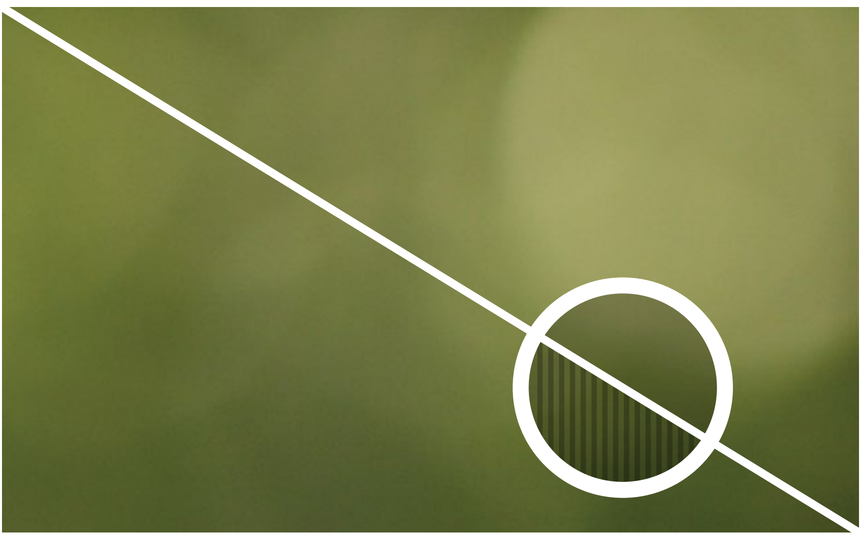
GEOMETRY ILLUMINATED

AN ILLUSTRATED INTRODUCTION TO EUCLIDEAN & NON-EUCLIDEAN GEOMETRY



MATTHEW HARVEY

[WORK IN PROGRESS - AUGUST 2012]

0. YOU CANT DEFINE EVERYTHING
AXIOMS AND MODELS

Axiom

Let's start with some very basic things. This book is about plane geometry, and in plane geometry you can't get much more basic than points and lines. So let's start there. The first thing to realize is that both of these things, points and lines, are abstractions. You will not find them in the real world. Oh sure, there are point-like things out there– atoms might be a good example. There are line-like things too– laser beams come to mind. But these physical manifestations fall short of "true" points and lines. Points and lines, in other words, are not things we can point to in the real world. In a casual setting, that may not be a big deal. After all, the whole of human experience requires us to deal with abstraction in a variety of contexts on a daily basis. But to try to develop a precise mathematical system from these abstractions– well, that is a little bit more problematic. Consider the opening statements in Euclid's *Elements*,

> Definition 1. A point is that which has no part.
> Definition 2. A line is breadthless length.

I have to admit, I do like those definitions. They are kind of poetic (at least as poetic as mathematics is permitted to be). But let's be honest– how much information do they really convey? Euclid doesn't define a part, nor does he define breadth or length. Were he to define those terms, they would be have to be described using other terms, which would in turn need their own definition, and so on. It isn't that Euclid's definitions are bad. It is that this is a hopeless situation. You can't define everything.

Modern geometry takes an entirely different approach to the issue of elementary definitions. In truth, I think it would be fair to say that modern geometry dodges the question. But it does so in such an artful way that you almost feel foolish for asking the question in the first place. Like its classical counterpart, modern geometry is built upon a foundation of a few basic terms, such as point and line. Unlike the classical approach, in modern geometry no effort is made to define those basic terms. In fact, they are called the *undefined*s of the system. Well, you may ask, what can I do with terms that have no meaning? This is where the *axioms* of the geometry come into play. All the behavior, properties and interactions of the undefined terms are described in a set of statements called the *axioms* of the system. No effort is made to argue for the truth of the axioms. How could you do so?– they are statements about terms which themselves have no meaning. As long as the axioms do not contradict one another, they will define some kind of geometry. It may be quite different from the Euclidean geometry to which we are accustomed, but it is a geometry none the less.

Model

Okay, you say, I see what you are saying, but I have done geometry before, and I really like those pictures and diagrams. They help me to understand what is going on. Well, I agree completely! Sure, a bad diagram can be misleading. Even a good diagram can be misleading at times. On the whole, though, I believe that diagrams lead more often than they mislead. The very thesis of this book is that illustrations are an essential part of the subject.

In that case, what is the relationship between illustrations and axioms? First of all, we have to accept that the illustrations are imperfect. Lines printed on paper have a thickness to them. They are finite in length. Points also have a length and width– otherwise we couldn't see them. That's just the way it has to be. But really, I don't think that is such a big deal. I think the focus on those imperfections tends to mask an even more important issue. And that is that these illustrations represent only one manifestation of the axioms. Points and lines as we depict them are one way to interpret the undefined terms of point and line. This intepretation happens to be consistent with all of the standard Euclidean axioms. But there may be a completely different interpretation of the undefineds which also satisfies the Euclidean axioms. Any such interpretation is called a *model* for the geometry. A geometry may have many models, and from a theoretical point of view, no one model is more right than any other. It is important, then, to prove facts about the geometry itself, and not peculiarities of one particular model.

Fano’s Geometry

To see how axiomatic geometry works without having our Euclidean intuition getting in the way, let’s consider a decidely non-Euclidean geometry called Fano’s geometry (named after the Italian algebraic geometer Gino Fano). In Fano’s geometry there are three undefined terms, *point*, *line*, and *on*. Five axioms govern these undefined terms.

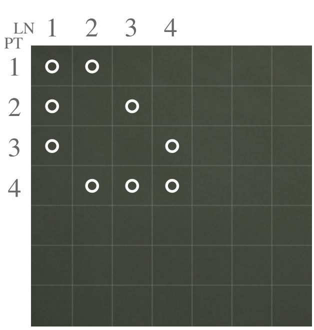
Ax 1. There exists at least one line.
Ax 2. There are exactly three points on each line.
Ax 3. Not all points are on the same line.
Ax 4. There is exactly one line on any two distinct points.
Ax 5. There is at least one point on any two distinct lines.

Fano’s geometry is a simple example of what is called a finite projective geometry. It is projective because, by the fifth axiom, all lines intersect one another (lines cannot be parallel). It is finite because, as we will see, it only contains finitely many points and lines. To get a sense of how an axiomatic proof works, let’s count the points and lines in Fano’s geometry.

THM
Fano’s geometry has exactly seven points and seven lines.

Proof. I have written this proof in the style I was taught in high school geometry, with a clear separation of each statement and its justification (in this case, an axiom). It is my understanding that geometry is rarely taught this way now. A shame, I think, since I think that this is a good way to introduce the idea of logical thought and proof.

1



This chart tracks the incidences of points on lines through the proof.

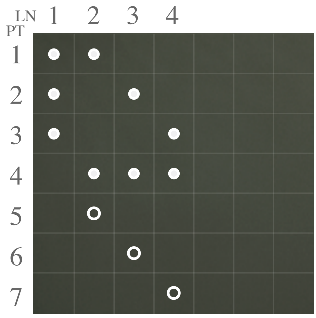
Ax 1 There is a line ℓ_1.

Ax 2 On ℓ_1, there are three points. Label them p_1, p_2 and p_3.

Ax 3 There is a fourth point p_4 that is not on ℓ_1.

Ax 4 There are lines: ℓ_2 on p_1 and p_4, ℓ_3 on p_2 and p_4, and ℓ_4 on p_3 and p_4. Each of these lines is distinct.

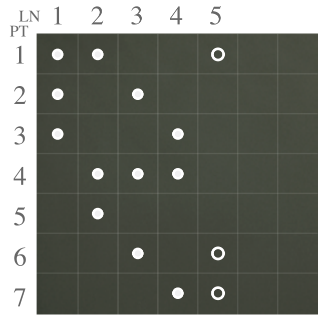
2



Ax 2 Each of these lines has a third point on it.

Ax 4 They are distinct and different from any of the previously declared points. Label them: p_5 on ℓ_2, p_6 on ℓ_3, and p_7 on ℓ_4.

3



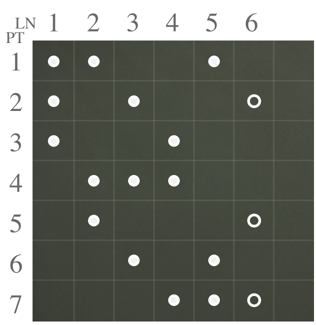
Ax 4 There must be a line ℓ_5 on p_1 and p_6.

Ax 2 The line ℓ_5 must have one more point on it.

Ax 4 That point cannot be either p_3 or p_4.

Ax 5 For ℓ_5 and ℓ_4 to intersect, the third point of ℓ_5 must be p_7.

4



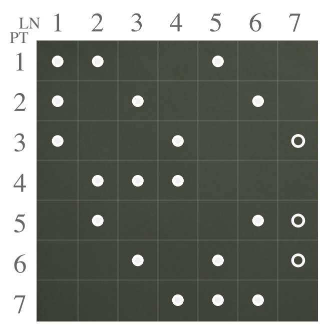
Ax 4 There must be a line ℓ_6 on p_2 and p_5.

Ax 2 The line ℓ_6 must have a third point on it.

Ax 4 That point cannot be p_3 or p_4.

Ax 5 For ℓ_6 and ℓ_4 to intersect, the third point of ℓ_6 must be p_7.

5



Ax 4 There must be a line ℓ_7 on p_3 and p_5.

Ax 2 The line ℓ_7 must have one more point on it.

Ax 4 That point cannot be p_2 or p_4.

Ax 5 For ℓ_7 and ℓ_3 to intersect, the third point of ℓ_7 must be p_6.

We now have seven points and seven lines as required. Could there be more? Let's suppose there were an eighth point p_8.

Ax 4 Then there would be a line ℓ_8 on p_1 and p_8.

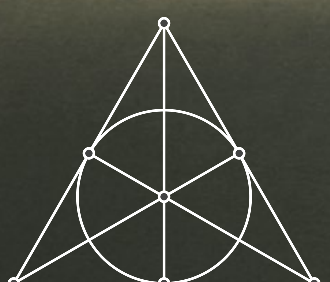
Ax 3 Line ℓ_8 would have to have another point on it.

Ax 4 This other point would have to be distinct from each of p_2 through p_7.

Ax 5 Then ℓ_8 would not share a point with ℓ_3 (and other lines as well). Thus there cannot be an eighth point.

Ax 4 There is now a line on every pair of points. Therefore there can be no more lines.

□



A model for Fano's geometry.
The nodes of the graph represent the points. The six segments and the circle represent the lines.

Further reading

Euclid's Elements is still a fantastic read. There are several editions available, both in text form and online, including, for instance, [3]. If you want to know more about projective geometry in general, I would recommend Coxeter's book [2]. For a finite projective planes, I have found a nice set of online notes by Jurgen Bierbrauer [1]. At the time of this writing they are available at the web address:

http://www.math.mtu.edu/∼jbierbra/HOMEZEUGS/finitegeom04.ps.

[1] Jürgen Bierbrauer. Finite geometries: MA5980. Lecture notes distributed on World Wide Web, 2004.

[2] H.S.M. Coxeter. *Projective Geometry*. Blaisdell Publishing Co., New York, 1st edition, 1964.

[3] Euclid. *The Thirteen Books of Euclid's Elements*. Dover Publications, New York, 2nd edition, 1956. Translated from the text of Heiberg, with introduction and commentary by Sir Thomas L. Heath.

1 NEUTRAL GEOMETRY

The goal of this book is to provide a pleasant but thorough introduction to Euclidean and non-Euclidean (hyperbolic) geometry. Before I go any further, let me clear up something that could lead to confusion on down the road. Some mathematicians use the term *non-Euclidean geometry* to mean any of a whole host of geometries which fail to be Euclidean for any number of reasons. The kind of non-Euclidean geometry that we will study in these lessons, and the kind that I mean when I use the term *non-Euclidean geometry*, is something much more specific– it is a geometry that satisfies all of Hilbert's axioms for Euclidean geometry except the parallel axiom.

It turns out that that parallel axiom is absolutely central to the nature of the geometry. The Euclidean geometry with the parallel axiom and the non-Euclidean geometry without it are radically different. Even so, Euclidean and non-Euclidean geometry are not polar opposites. As different as they are in many ways, they still share many basic characteristics. Neutral geometry (also known as absolute geometry in older texts) is the study of those commonalities.

1. OUR DUCKS IN A ROW
THE AXIOMS OF INCIDENCE
AND ORDER

From Euclid to Hilbert

You pretty much have to begin a study of Euclidean geometry with at least some mention of Euclid's *Elements*, the book that got the ball rolling over two thousand years ago. *The Elements* opens with a short list of definitions. As discussed in the previous chapter, the first few of these definitions are a little problematic. If we can push past those, we get to Euclid's five postulates, the core accepted premises of his development of the subject.

EUCLID'S POSTULATES

P1 To draw a straight line from any point to any point.

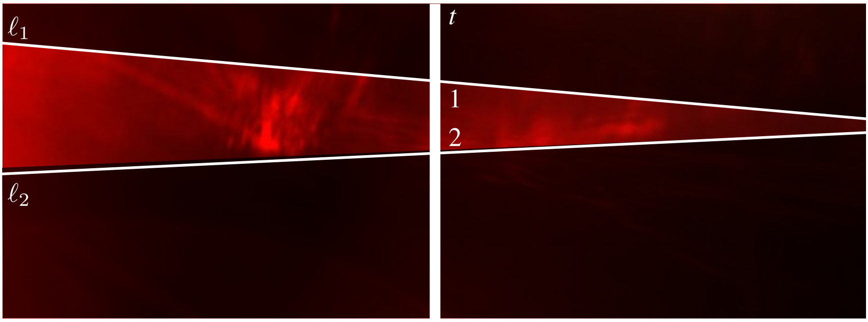
P2 To produce a finite straight line continuously in a straight line.

P3 To describe a circle with any center and distance.

P4 That all right angles are equal to one another.

P5 That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The first three postulates describe constructions. Today we would probably reinterpret them as statements about the existence of certain objects. The fourth provides a way to compare angles. As for the fifth, well, in all of history, not many sentences have received as much scrutiny as that one.



Euclid's Parallel Postulate

Because $(\angle 1) + (\angle 2) < 180^\circ$, ℓ_1 *and* ℓ_2 *intersect on this side of* t.

When you look at these postulates, and Euclid's subsequent development of the subject from them, it appears that Euclid may have been attempting an axiomatic development of the subject. There is some debate, though, about the extent to which Euclid really was trying to do that. His handling of "S·A·S," for example, is not founded upon the postulates, and not merely in a way that might be attributed to oversight. With a couple thousand years between us and him, we can only guess at his true intentions. In any case, Euclidean geometry was not properly and completely axiomatized until much later, at the end of the nineteenth century by the German mathematician David Hilbert. His 1899 book, *The Foundations of Geometry* gave an axiomatic description of what we think of as Euclidean geometry. Subsequently, there have been several other axiomatizations, including notably ones by Birkhoff and Tarski. The nice thing about Hilbert's approach is that proofs developed in his system "feel" like Euclid's proofs. Some of the other axiomatizations, while more streamlined, do not retain that same feel.

Neutral Geometry

It might be an obvious statement, but it needs to be said: Euclid's Fifth Postulate does not look like the other four. It is considerably longer and more convoluted than the others. For that reason, generations of geometers after Euclid hoped that the Fifth might actually be provable– that it could be taken as a theorem rather than a postulate. From their efforts (which, by the way, were unsuccessful) there arose a whole area of study. Called *neutral geometry* or *absolute geometry*, it is the study of the geometry of the plane without Euclid's Fifth Postulate.

So what exactly do you give up when you decide not to use Euclid's Fifth? Essentially Euclid's Fifth tells us something about the nature of parallel lines. It does so in a rather indirect way, though. Nowadays it is common to use Playfair's Axiom in place of Euclid's Fifth because it addresses the issue of parallels much more directly. Playfair's Axiom both implies and is implied by Euclid's Fifth, so the two statements can be used interchangeably.

PLAYFAIR'S AXIOM
For any line ℓ and for any point P which is not on ℓ, there is exactly one line through P which is parallel to ℓ.

Even without Playfair's Axiom, it is relatively easy to show that there must be at *least* one parallel through P, so what Playfair's Axiom is really telling us is that in Euclidean geometry there cannot be *more* than one parallel. The existence of a unique parallel is crucial to many of the proofs of Euclidean geometry. Without it, neutral geometry is quite limited. Still, neutral geometry is the common ground between Euclidean and non-Euclidean geometries, and it is where we begin our study.

In the first part of this book, we are going to develop neutral geometry following the approach of Hilbert. In Hilbert's system there are five undefined terms: *point*, *line*, *on*, *between*, and *congruent*. Fifteen of his axioms are needed to develop neutral plane geometry. Generally the axioms are grouped into categories to make it a bit easier to keep track of them: the axioms of incidence, the axioms of order, the axioms of congruence, and the axioms of continuity. We will investigate them in that order over the next several chapters.

Incidence

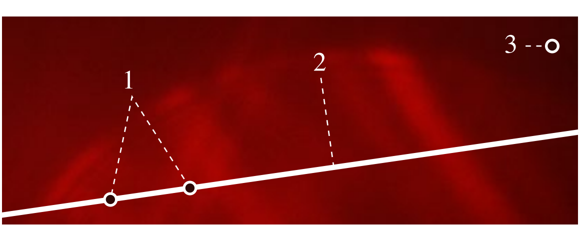
Hilbert's first set of axioms, the axioms of incidence, describe the interaction between points and lines provided by the term *on*. *On* is a binary relationship between points and lines so, for instance, you can say that a point P is (or is not) on a line ℓ. In situations where you want to express the line's relationship to a point, rather than saying that a line ℓ is on a point P (which is technically correct), it is much more common to say that ℓ passes through P.

THE AXIOMS OF INCIDENCE

In 1 There is a unique line on any two distinct points.

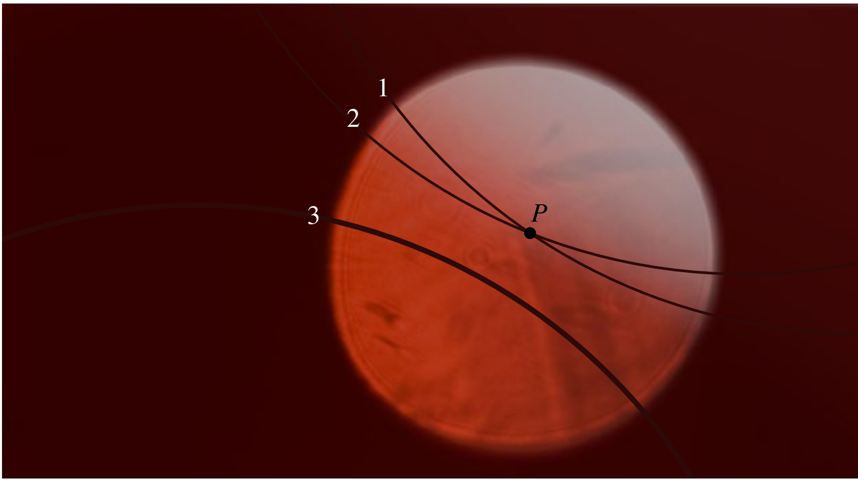
In 2 There are at least two points on any line.

In 3 There exist at least three points that do not all lie on the same line.



Incidence

1 Two points on a line
2 A line on two points
3 And there's more.



Lines 1 and 2 intersect. Both are parallel to line 3. Because there appear to be two lines through P parallel to line 3, this does not look like Euclidean geometry.

By themselves, the axioms of incidence do not afford a great wealth of theorems. Some notation and a few definitions are all we get. First, the notation. Because of the first axiom, there is only one line through any two distinct points. Therefore, for any two distinct points A and B, we use the notation $\leftarrow AB \rightarrow$ to denote the line through A and B. As you are probably all aware, this is not exactly the standard notation for a line. Conventionally, the line symbol is placed above the points. I just don't like that notation in print– unless you have lots of room between lines of text, the symbol crowds the line above it.

Now the definitions. Any two distinct points lie on one line. Three or more points may or may not all lie on the same line.

DEF: COLINEARITY

Three or more points are *colinear* if they are all on the same line and are *non-colinear* if they are not.

According to the first axiom, two lines can share at most one point. However, they may not share any points at all.

DEF: PARALLEL AND INTERSECTING

Two lines *intersect* if there is a point P which is on both of them. In this case, P is the *intersection* or *point of intersection* of them. Two lines which do not share a point are *parallel*.

Order

The axioms of order describe the undefined term *between*. Between is a relation between a point and a pair of points. We say that a point B is, or is not, between two points A and C and we use the notation $A * B * C$ to indicate that B is between A and C. Closely related to this "between-ness" is the idea that a line separates the plane. This behavior, which is explained in the last of the order axioms, depends upon the following definition.

DEF: SAME SIDE

Let ℓ be a line and let A and B be two points which are not on ℓ. Points A and B are on the *same side* of ℓ if either ℓ and $\overleftrightarrow{AB}$ do not intersect at all, or if do they intersect but the point of intersection is not between A and B.

So now, without further delay, the Axioms of Order describing the properties of between.

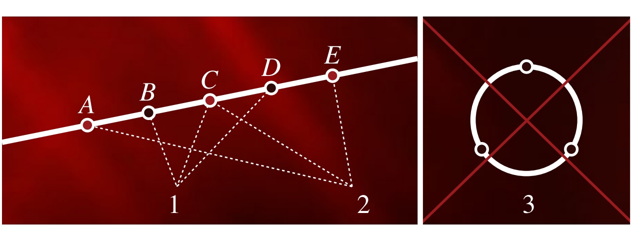
THE AXIOMS OF ORDER

Or 1 If $A * B * C$, then the points A, B, C are distinct colinear points, and $C * B * A$.

Or 2 For any two points B and D, there are points A, C, and E, such that $A * B * D$, $B * C * D$ and $B * D * E$.

Or 3 Of any three distinct points on a line, exactly one lies between the other two.

Or 4 *The Plane Separation Axiom.* For any line ℓ and points A, B, and C which are not on ℓ: (i) If A and B are on the same side of ℓ and A and C are on the same side of ℓ, then B and C are on the same side of ℓ. (ii) If A and B are not on the same side of ℓ and A and C are not on the same side of ℓ, then B and C are on the same side of ℓ.



Order

1 Points in order
2 Between and beyond
3 But no circularity

The last of these, the Plane Separation Axiom (*PSA*), is a bit more to digest than the previous axioms. It is pretty critical though– it is the axiom which limits plane geometry to two dimensions. Let's take a closer look. Let ℓ be a line and let P be a point which is not on ℓ. We're going to define two sets of points.

S_1: P itself and all points on the same side of ℓ as P.

S_2: all points which are not on ℓ nor on the same side of ℓ as P

By the second axiom of order both S_1 and S_2 are nonempty sets. The first part of *PSA* tells us is that all the points of S_1 are on the same side; the second part tells us that all the points of S_2 are on the same side. Hence there are two and only two sides to a line. Because of this, we can refer to points which are not on the same side of a line as being on *opposite sides*.

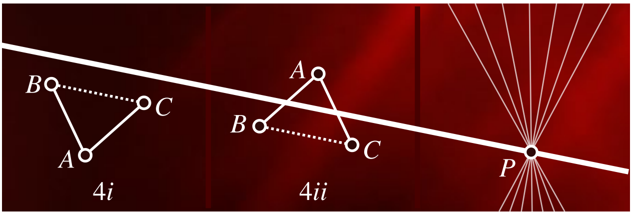
Just as a line separates the remaining points of the plane, a point on a line separates the remaining points on that line. If P is between A and B, then A and B are on *opposite sides* of P. Otherwise, A and B are on the *same side* of P. You might call this separation of a line by a point "line separation". It is a direct descendent of plane separation via the following simple correspondence. For three distinct points A, B, and P on a line ℓ,

A, B on the same side of P	$\Longleftrightarrow$	A, B are on the same side of any line through P other than ℓ
A, B on opposite sides of P	$\Longleftrightarrow$	A, B are on opposite sides of any line through P other than ℓ

Because of this, there is a counterpart to the Plane Separation Axiom for lines. Suppose that A, B, C and P are all on a line. (1) If A and B are on the same side of P and A and C are on the same side of P, then B and C are on the same side of P. (2) If A and B are on opposite sides of P and A and C are on opposite sides of P, then B and C are on the same side of P. As a result, a point divides a line into *two* sides.

PSA

A line separates the plane. A point separates a line.



With *between*, we can now introduce some a few of the main characters in this subject.

DEF: LINE SEGMENT
For any two points A and B, the *line segment* between A and B is the set of points P such that $A * P * B$, together with A and B themselves. The points A and B are called the *endpoints* of the segment.

DEF: RAY
For two distinct points A and B, the *ray* from A through B consists of the point A together with all the points on $\leftarrow AB \rightarrow$ which are on the same side of A as B. The point A is called the *endpoint* of the ray.

The notation for the line segment between A and B is AB. For rays, I write $AB\!\rightarrow$ for the ray with endpoint A through the point B. As with my notation for lines, this is a break from the standard notation which places the ray symbol above the letters.

DEF: OPPOSITE RAY
For any ray $AB \rightarrow$, the opposite ray $(AB \rightarrow)^{op}$ consists of the point A together with all the points of $\leftarrow AB \rightarrow$ which are on the opposite side of A from B.

Putting Points in Order

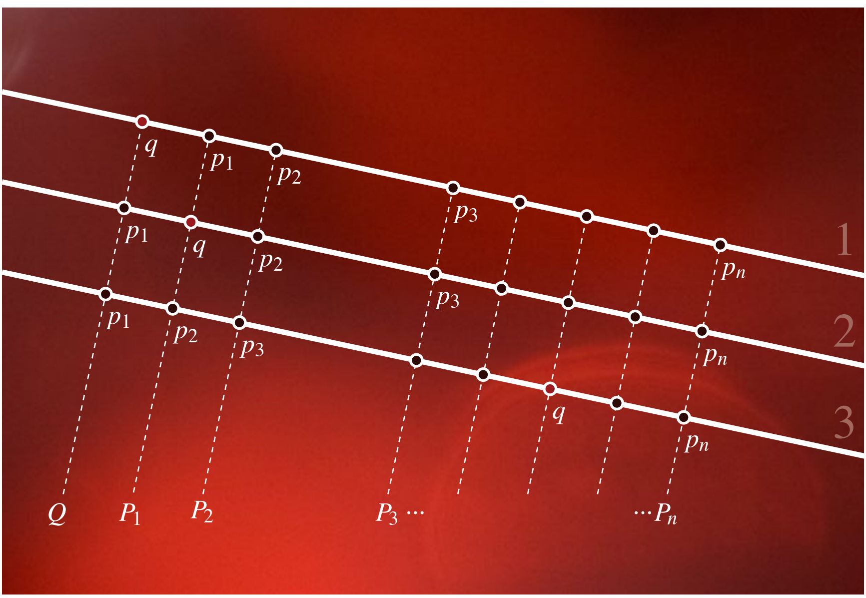
The order axioms describe how to put three points in order. Sometimes, though, three is not enough. It would be nice to know that more than three points on a line can be ordered in a consistent way. Thankfully, the axioms of order make this possible as well.

THM: ORDERING POINTS
Given $n \geq 3$ colinear points, there is a labeling of them $P_1, P_2, \ldots, P_n$ so that if $1 \leq i < j < k \leq n$, then $P_i * P_j * P_k$. In that case, we write

$$P_1 * P_2 * \cdots * P_n.$$

Proof. This is a proof by induction. The initial case, when there are just three points to put in order, is an immediate consequence of the axioms of order. Now let's assume that any set of n colinear points can be put in order, and let's suppose we want to put a set of $n+1$ colinear points in order. I think the natural way to do this is to isolate the first point (call it Q), put the remaining points in order, and then stick Q back on the front. The problem with this approach is that figuring out which point is the first point essentially presupposes that you *can* put the points in order. Getting around this is a little delicate, but here's how it works. Choose n of the $n+1$ points. Put them in order and label them so that $p_1 * p_2 * \cdots * p_n$. Let q be the one remaining point. Now, one of the following three things must happen:

$$q * p_1 * p_2 \quad \text{or} \quad p_1 * q * p_2 \quad \text{or} \quad p_1 * p_2 * q.$$

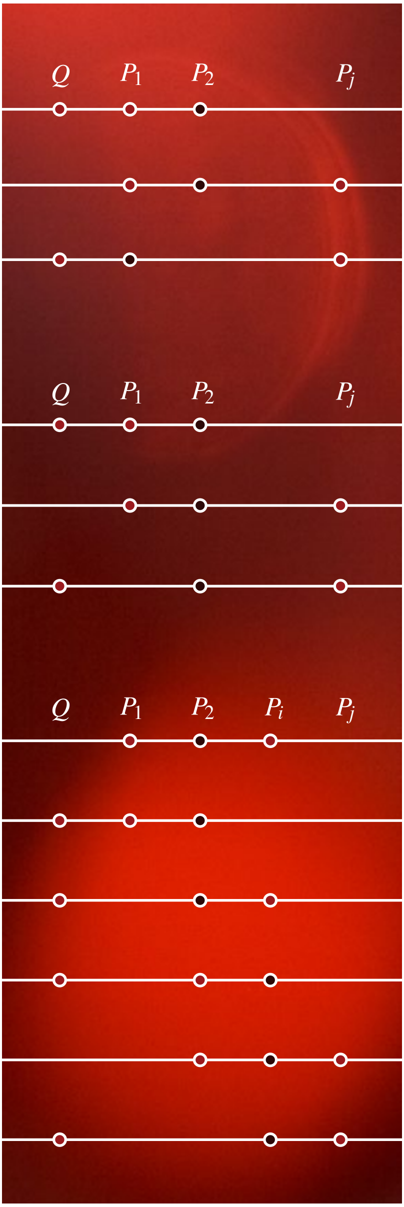


The three possible positions of q in relation to p_1 and p_2.

In the first case, let $Q = q$ and let $P_1 = p_1$, $P_2 = p_2$, $\ldots$, $P_n = p_n$. In the second and third cases, let $Q = p_1$.Then put the remaining points p_1, $\ldots$, p_n and q in order and label them $P_1, P_2, \ldots, P_n$. Having done this, we have two pieces of an ordering

$$Q * P_1 * P_2 \quad \text{and} \quad P_1 * P_2 * \cdots * P_n.$$

The proof is not yet complete, though, because we still need to show that Q is ordered properly with respect to the remaining P's. That is, we need to show $Q * P_i * P_j$ when $1 \leq i < j \leq n$. Let's do that (in several cases).



Case 1: $i = 1$.
The result is given when $j = 2$, so let's suppose that $j > 2$. Then:

1. $Q * P_1 * P_2$ so Q and P_1 are on the same side of P_2.
2. $P_1 * P_2 * P_j$ so P_1 and P_j are on opposite sides of P_2.
3. Therefore Q and P_j are on opposite sides of P_1, so $Q * P_1 * P_j$.

Case 2: $i = 2$.

1. $Q * P_1 * P_2$ so Q and P_1 are on the same side of P_2.
2. $P_1 * P_2 * P_j$ so P_1 and P_j are on opposite sides of P_2.
3. Therefore Q and P_j are on opposite sides of P_2, so $Q * P_2 * P_j$.

Case 3: $i > 2$.

1. $P_1 * P_2 * P_i$ so P_1 and P_i are on opposite sides of P_2.
2. $Q * P_1 * P_2$ so Q and P_1 are on the same side of P_2.
3. Therefore Q and P_i are on opposite sides of P_2, so $Q * P_2 * P_i$.
4. Consequently, Q and P_2 are on the same side of P_i.
5. Meanwhile, $P_2 * P_i * P_j$ so P_2 and P_j are on opposite sides of P_i.
6. Therefore, Q and P_j are on opposite sides of P_i, so $Q * P_i * P_j$.

$\sqcap$

Exercises

1. Prove that if $A * B * C$ then $AB \subset AC$ and $AB\rightarrow \subset AC\rightarrow$.

2. Prove that if $A * B * C * D$ then $AC \cup BD = AD$ and $AC \cap BD = BD$.

3. Prove that the points which are on both $AB\rightarrow$ and $BA\rightarrow$ are the points of AB.

4. Use the axioms of order to show that there are infinitely many points on any line and that there are infinitely many lines through a point.

5. The familiar model for Euclidean geometry is the "Cartesian model." In that model, points are interepreted as coordinate pairs of real numbers (x, y). Lines are loosely interpreted as equations of the form

$$Ax + By = C$$

but technically, there is a little bit more to it than that. First, A and B cannot both simultaneously be zero. Second, if $A' = kA$, $B' = kB$, and $C' = kC$ for some nonzero constant k, then the equations $Ax + By = C$ and $A'x + B'y = C'$ both represent the same line [in truth then, a line is represented by an equivalence class of equations]. In this model, a point (x, y) is on a line $Ax + By = C$ if its coordinates make the equation true. With this interpretation, verify the axioms of incidence.

6. In the Cartesian model, a point (x_2, y_2) is between two other points (x_1, y_1) and (x_3, y_3) if:

   1. the three points are distinct and on the same line, and
   2. x_2 is between x_1 and x_3 (either $x_1 \leq x_2 \leq x_3$ or $x_1 \geq x_2 \geq x_3$), and
   3. y_2 is between y_1 and y_3 (either $y_1 \leq y_2 \leq y_3$ or $y_1 \geq y_2 \geq y_3$).

   With this interpretation, verify the axioms of order.

Further reading

For these first few“moves”, we are pretty constricted, with few results to build from and very little flexibility about where we can go next. Since we have adopted the axioms of Hilbert, our initial steps (in this and the next few lessons) follow fairly closely those of Hilbert in his *Foundations of Geometry* [2].

In addition, let me refer you to a few more contemporary books which examine the first steps in the development of the subject. Moise's *Elementary Geometry from an Advanced Standpoint* [3] is one of my favorites. I have taught from both Wallace and West's *Roads to Geometry* [4], and Greenberg's *Euclidean and Non-Euclidean Geometries* [1].

[1] Marvin J. Greenberg. *Euclidean and Non-Euclidean Geometries: Development and History*. W.H. Freeman and Company, New York, 4th edition, 2008.

[2] David Hilbert. *The Foundations of Geometry*.

[3] Edwin E. Moise. *Elementary Geometry from an Advanced Standpoint*. Addison Wesley Publishing Company, Reading, Massachusetts, 2nd edition, 1974.

[4] Edward C. Wallace and Stephen F. West. *Roads to Geometry*. Pearson Education, Inc., Upper Saddle River, New Jersey, 3rd edition, 2004.

2. IN ONE SIDE, OUT THE OTHER

ANGLES AND TRIANGLES

These are the first steps. They are tentative. But it is right to be cautious. It is so difficult keeping intuition from making unjustified leaps. The two main theorems in this lesson, Pasch's Lemma and the Crossbar Theorem, are good examples of this. Neither can be found in Euclid's Elements. They just seem so obvious that I guess it didn't occur to him that they needed to be proved (his framework of postulates would not allow him to prove those results anyway). The kind of intersections that they guarantee are essential to many future results, though, so we must not overlook them.

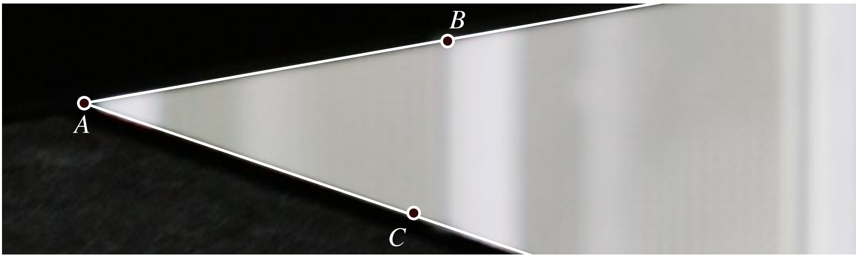
Angles and Triangles

In the last lesson we defined ray and segment. They are the most elementary of objects, defined directly from the undefined terms. Now in this lesson, another layer: angles and triangles, which are built from rays and segments.

DEF: ANGLE
An *angle* consists of a (unordered) pair of non-opposite rays with the same endpoint. The mutual endpoint is called the *vertex* of the angle.

Let's talk notation. If the two rays are $AB\rightarrow$ and $AC\rightarrow$, then the angle they form is written $\angle BAC$, with the endpoint listed in the middle spot. There's more than one way to indicate that angle though. For one, it does not matter which order the rays are taken, so $\angle CAB$ points to the same angle as $\angle BAC$. And if B' is on $AB\rightarrow$ and C' is on $AC\rightarrow$ (not the endpoint of course), then $\angle B'AC'$ is the same as $\angle BAC$ too. Frequently, it is clear in the problem that you only care about one angle at a particular vertex. On those occasions you can often get away with the abbreviation $\angle A$ in place of the full $\angle BAC$. Just as a line divides the plane into two sides, so too does an angle. In this case the two parts are the interior and the exterior of the angle.

DEF: ANGLE INTERIOR
A point lies *in the interior* or is an *interior point* of $\angle BAC$ if it is on the same side of $\leftarrow AB\rightarrow$ as C and same side of $\leftarrow AC\rightarrow$ as B. A point which does not lie in the interior of the angle and does not lie on either of the rays composing the angle is *exterior* to the angle and is called an *exterior point*.



$\angle BAC$. *The light region is the interior. The dark the exterior.*

The last definition in this section is that of the triangle. Just as an angle is formed by joining two rays at their mutual endpoint, a triangle is formed by joining three segments at mutual endpoints.

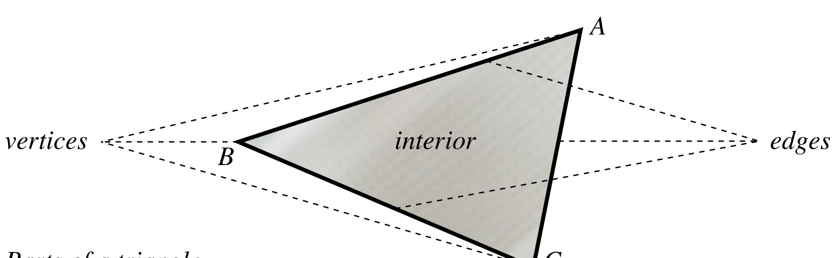
DEF: TRIANGLE

A *triangle* is an (unordered) triple of non-colinear points and the points on the segments between each of the three pairs of points. Each of the three points is called a *vertex* of the triangle. Each of the three segments is called a *side* or *edge* of the triangle.

If A, B, and C are non-colinear points then we write $\triangle ABC$ for the triangle. The ordering of the three vertices does not matter, so there is more than one way to write a given triangle:

$$\triangle ABC = \triangle ACB = \triangle BAC = \triangle BCA = \triangle CAB = \triangle CBA.$$

The three sides of $\triangle ABC$ are AB, AC, and BC. The three angles $\angle ABC$, $\angle BCA$ and $\angle CAB$ are called the *interior angles* of $\triangle ABC$. A point which is in the interior of all the three of the interior angles is said to be *inside* the triangle. Together they form the *interior* of the triangle. Points which are not inside the triangle and aren't on the triangle itself, are said to be *outside* the triangle. They make the *exterior* of the triangle.



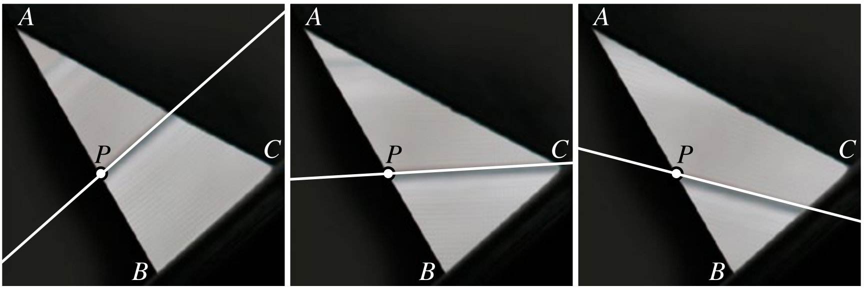
Parts of a triangle

A Line Passes Through It

The rest of this lesson is dedicated to three fundamental theorems. The first, a result about lines crossing triangles is called Pasch's Lemma after Moritz Pasch, a nineteenth century German mathematician whose works are a precursor to Hilbert's. It is a direct consequence of the Plane Separation Axiom. The second result, the Crossbar Theorem, is a bit more difficult. It deals with lines crossing through the vertex of an angle. The third says that rays with a common endpoint can be ordered in a consistent way, in the same way that points on a line can be ordered.

> PASCH'S LEMMA
> If a line intersects a side of a triangle at a point other than a vertex, then it must intersect another side of the triangle. If a line intersects all three sides of a triangle, then it must intersect two of the sides at a vertex.

Proof. Suppose that a line ℓ intersects side AB of $\triangle ABC$ at a point P other than the endpoints. If ℓ also passes through C, then that's the other intersection; in this case ℓ does pass through all three sides of of the triangle, but it passes through two of them at a vertex. Now what if ℓ does not pass through C? There are only two possibilities: either C is on the same side of ℓ as A, or it is on the opposite side of ℓ from A. This is where the Plane Separation Axiom comes to the rescue. Because P is between A and B, those two points have to be on opposite sides of ℓ. Thus, if C is on the same side of ℓ as A, then it is on the opposite side of ℓ from B, and so ℓ intersects BC but not AC. On the other hand, if C is on the opposite side of ℓ from A, then it is on the same side of ℓ as B, so ℓ intersects AC but not BC. Either way, ℓ intersects two of the three sides of the triangle. $\square$



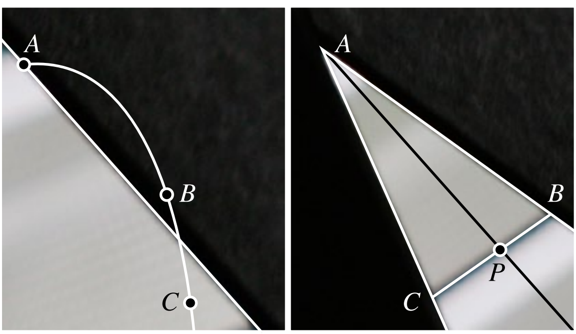
ℓ passes through AC *ℓ passes through C* *ℓ passes through BC*

As I mentioned at the start of the section, the proof of the Crossbar Theorem is more challenging. I think it is helpful to separate out one small part into the following lemma.

LEMMA

If A is a point on line ℓ, and B is a point which is not on ℓ, then all the points of $AB\rightarrow$ (and therefore all the points of AB) except A are on the same side of ℓ as B.

Proof. If C is any point on $AB\rightarrow$ other than A or B, then C has to be on the same side of A as B, and so either $A * B * C$ or $A * C * B$. Either way, $\leftarrow AC \rightarrow$ and ℓ intersect at the point A, but that point of intersection does not lie between B and C. Hence B and C are on the same side of ℓ. $\square$



(l) The lemma says that a ray cannot recross a line like this. (r) The Crossbar Theorem guarantees the existence of the point P.

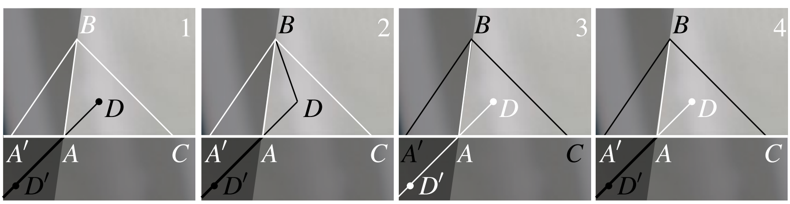
THE CROSSBAR THEOREM

If D is an interior point of angle $\angle BAC$, then the ray $AD\rightarrow$ intersects the segment BC.

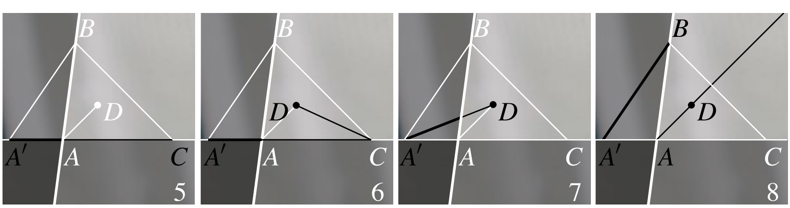
Proof. If you take a couple minutes to try to prove this for yourself, you will probably find yourself thinking– hey, this seems awfully similar to Pasch's Lemma– we could use $\triangle ABC$ for the triangle and $\leftarrow AD \rightarrow$ for the line. The problem is that one pesky condition in Pasch's Lemma: the given intersection of the line and the triangle can't be at a vertex. In the situation we have here, the ray in question $AD \rightarrow$ *does* pass through the vertex. Still, the basic idea is sound. The actual proof does use Pasch's Lemma, we just have to bump the triangle a little bit so that $AD\rightarrow$ doesn't cross through the vertex.

According to the second axiom of order, there are points on the opposite side of A from C. Let A' be one of them. Now $\leftarrow AD \rightarrow$ intersects the side $A'C$ of the triangle $\triangle A'BC$. By Pasch's Lemma, $\leftarrow AD \rightarrow$ must intersect one of the other two sides of triangle, either $A'B$ or BC. There are two scenarios to cause concern. First, what if $\leftarrow AD \rightarrow$ crosses $A'B$ instead of BC? And second, what if $\leftarrow AD \rightarrow$ does cross BC, but the intersection is on $(AD\rightarrow)^{op}$ instead of $AD\rightarrow$ itself?

I think it is easier to rule out the second scenario first so let's start there. (1)If D' is any point on $(AD\rightarrow)^{op}$, then it is on the opposite side of A from D. Therefore D' and D are on opposite sides of $\leftarrow A'C \rightarrow$. (2)Since D is an interior point, it is on the same side of $\leftarrow A'C \rightarrow$ as B, and so D' and B are on opposite sides of $A'C$. (3)By the previous lemma, all the points of $A'B$ and of BC are on the same side of $\leftarrow A'C \rightarrow$ as B. (4)Therefore they are on the opposite side of $\leftarrow A'C \rightarrow$ fom D', so no point of $(AD\rightarrow)^{op}$ may lie on either $A'B$ or BC.

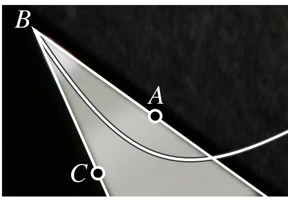


With the opposite ray ruled out entirely, we now just need to make sure that $AD\rightarrow$ does not intersect $A'B$. (5)Points A' and C are on opposite sides of $\leftarrow AB \rightarrow$. (6)Because D is an interior point, D and C are on the same side of $\leftarrow AB \rightarrow$. (7)Therefore A' and D are on opposite sides of $\leftarrow AB \rightarrow$. (8)Using the preceding lemma, all the points of $A'B$ are on opposite sides of $\leftarrow AB \rightarrow$ from all the points of $AD\rightarrow$. This means that $AD\rightarrow$ cannot intersect $A'B$, so it must intersect BC. □



The Crossbar Theorem provides a essential conduit between the notion of *between* for points and *interior* for angles. I would like to use that conduit in the next theorem, which is the angle interior analog to the ordering of points theorem in the last lesson. First let me state a useful lemma.

LEMMA 2
Consider an angle $\angle ABC$ and a ray r whose endpoint is B. Either all the points of r other than B lie in the interior of $\angle ABC$, or none of them do.



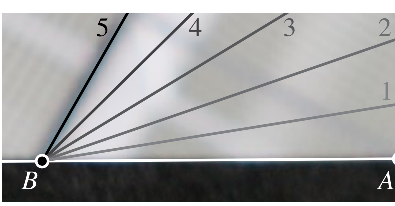
Lemma 2. Rays cannot do this.

I am going to leave the proof of this lemma to you, the reader. It is a relatively straightforward proof, and lemma 1 should provide some useful clues. Now on to the theorem.

THM: ORDERING RAYS
Consider $n \geq 2$ rays with a common basepoint B which are all on the same side of a line $\overleftrightarrow{AB}$ through B. There is an ordering of them:

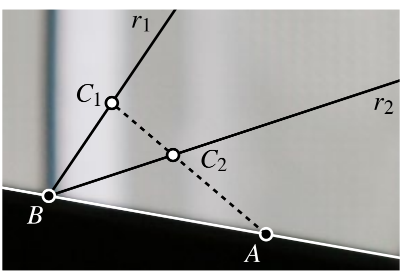
$$r_1,\ r_2,\ \ldots, r_n$$

so that if $i < j$ then r_i is in the interior of the angle formed by $\overrightarrow{BA}$ and r_j.



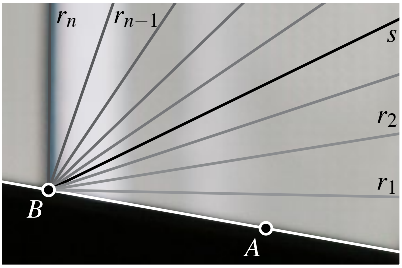
An ordering of five rays and five angles so that each ray is in the interior of all of the subsequent angles.

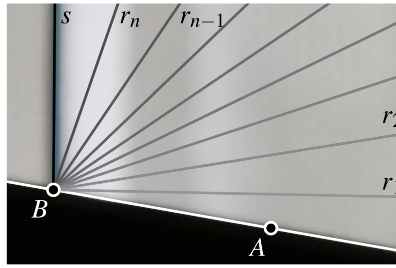
Proof. I am going to use a proof by induction. First consider the case of just $n = 2$ rays, r_1 and r_2. If r_1 lies in the interior of the angle formed by $BA\rightarrow$ and r_2, then we've got it. Let's suppose, though, that r_1 does not lie in the interior of that angle. There are two requirements for r_1 to lie in the interior: (1) it has to be on the same side of $\leftarrow AB\rightarrow$ as r_2 and (2) it has to be the same side of r_2 as A. From the very statement of the theorem, we can see that r_1 has to satisfy the first requirement, so if r_1 is not in the interior, the problem has got to be with the second requirement. That means that any point C_1 on r_1 has to be on the opposite side of r_2 from A– that is, the line containing r_2 must intersect AC_1. Actually we can be a little more specific about where this intersection occurs: you see, AC_1 and r_2^{op} are on opposite sides of $\leftarrow AB\rightarrow$ so they cannot intersect. Therefore the intersection is not on r_2^{op}– it has to be on r_2 itself. Call this intersection point C_2. Then $A * C_2 * C_1$ so C_2 is on the same side of r_1 as A. Therefore r_2 is on the same side of r_1 as A, and so r_2 is in the interior of the angle formed by $BA\rightarrow$ and r_1. Then it is just a matter of switching the labeling of r_1 and r_2 to get the desired result.



The base case: what happens if r_1 is not in the interior of the angle formed by $BA\rightarrow$ and r_2?

Now let's tackle the inductive step. Assume that any n rays can be put in order and consider a set of $n+1$ rays all sharing a common endpoint B and on the same side of the line $\leftarrow AB\rightarrow$. Take n of those rays and put them in order as $r_1, r_2, \ldots, r_n$. That leaves just one more ray– call it s. What I would like to do is to compare s to what is currently the "outermost" ray, r_n. One of two things can happen: either [1] s lies in the interior of the angle formed by $BA\rightarrow$ and r_n, or [2] it doesn't, and in this case, as we saw in the proof of the base case, that means that r_n lies in the interior of the angle formed by $BA\rightarrow$ and s. Our path splits now, as we consider the two cases.



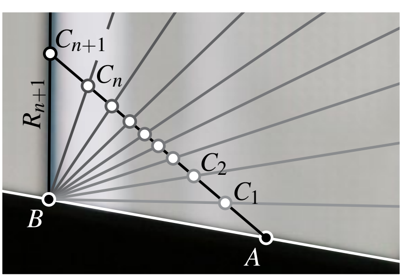


[1] Here r_n is the outermost ray, so let's relabel it as R_{n+1}. The remaining rays r_1, r_2, ..., r_{n-1} and s are all in the interior of the angle formed by $BA\rightarrow$ and R_{n+1}. Therefore, if C_{n+1} is any point on R_{n+1} (other than B) then each of r_1, r_2, ..., r_{n-1} and s intersect the segment AC_{n+1} (this is the Crossbar Theorem in action). We can put all of those intersection points in order

$$A * C_1 * C_2 * \cdots * C_n * C_{n+1}.$$

[2] In this case, we will eventually see that s is the outermost ray, but all we know at the outset is that it is farther out than r_n. Let's relabel s as R_{n+1} and let C_{n+1} be a point on this ray. Since r_n is in the interior of the angle formed by $BA\rightarrow$ and R_{n+1}, by the Crossbar Theorem, r_n must intersect AC_{n+1}. Let C_n be this intersection point. But we know that $r_1, r_2, \ldots, r_{n-1}$ lie in the interior of the angle formed by $BA\rightarrow$ and R_n, so AC_n must intersect each of $r_1, r_2, \ldots, r_n$. We can put all of those intersection points in order

$$A * C_1 * C_2 * \cdots * C_n * C_{n+1}.$$



Once the outermost ray is identified, a line connecting that ray to A intersects all the other rays (because of the Crossbar Theorem).

With the rays sorted and the intersections marked, the two strands of the proofs merge. Label the ray with point C_i as R_i. Then, for any $i < j$, C_i is on the same side of C_j as A, and so R_i is in the interior of the angle formed by $BA\rightarrow$ and C_j. This is the ordering that we want. □

Exercises

1. Prove that there are points in the interior of any angle. Similarly, prove that there are points in the interior of any triangle.

2. Suppose that a line ℓ intersects a triangle at two points P and Q. Prove that all the points on the segment PQ other than the endpoints P and Q are in the interior of the triangle.

3. We have assumed Plane Separation as an axiom and used it to prove Pasch's Lemma. Try to reverse that– in other words, assume Pasch's Lemma and prove the Plane Separation Axiom.

4. Let P be a point in the interior of $\angle BAC$. Prove that all of the points of $AP\rightarrow$ other than A are also in the interior of $\angle BAC$. Prove that none of the points of $(AP\rightarrow)^{op}$ are in the interior of $\angle BAC$.

5. Prove Lemma 2.

6. A model for a non-neutral geometry: $\mathbb{Q}^2$. We alter the standard Euclidean model $\mathbb{R}^2$ so that the only points are those with rational coordinates. The only lines are those that pass through at least two rational points. Incidence and order are as in the Euclidean model. Demonstrate that this models a geometry which satisfies all the axioms of incidence and order except the Plane Separation Axiom. Show that Pasch's Lemma and the Crossbar Theorem do not hold in this geometry.

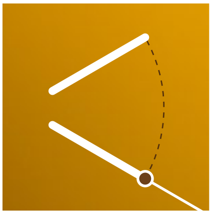
References

I got my proof of the Crossbar Theorem from Moise's book on Euclidean geometry [1].

[1] Edwin E. Moise. *Elementary Geometry from an Advanced Standpoint*. Addison Wesley Publishing Company, Reading, Massachusetts, 2nd edition, 1974.

3. CONGRUENCE VERSE I

OBJECTIVE: SAS AND ASA

Cg1 *The Segment Construction Axiom* If A and B are distinct points and if A' is any point, then for each ray r with endpoint A', there is a unique point B' on r such that $AB \simeq A'B'$.

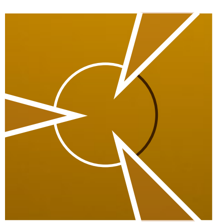
Cg2 Segment congruence is reflexive (every segment is congruent to itself), symmetric (if $AA' \simeq BB'$ then $BB' \simeq AA'$), and transitive (if $AA' \simeq BB'$ and $BB' \simeq CC'$, then $AA' \simeq CC'$).

Cg3 *The Segment Addition Axiom* If $A * B * C$ and $A' * B' * C'$, and if $AB \simeq A'B'$ and $BC \simeq B'C'$, then $AC \simeq A'C'$.

Cg4 *The Angle Construction Axiom* Given $\angle BAC$ and any ray $\overrightarrow{A'B'}$, there is a unique ray $\overrightarrow{A'C'}$ on a given side of the line $\overleftrightarrow{A'B'}$ such that $\angle BAC \simeq \angle B'A'C'$.

Cg5 Angle congruence is reflexive (every angle is congruent to itself), symmetric (if $\angle A \simeq \angle B$, then $\angle B \simeq \angle A$), and transitive (if $\angle A \simeq \angle B$ and $\angle B \simeq \angle C$, then $\angle A \simeq \angle C$).

Cg6 *The Side Angle Side (S·A·S) Axiom*. Consider two triangles: $\triangle ABC$ and $\triangle A'B'C'$. If $AB \simeq A'B'$, $\angle B \simeq \angle B'$, and $BC \simeq B'C'$, then $\angle A \simeq \angle A'$.

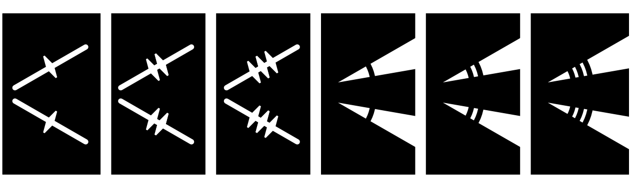
  

I think this is the lesson where the geometry we are doing starts to look like the geometry you know. I don't think your typical high school geometry class covers Pasch's Lemma or the Crossbar Theorem, but I'm pretty sure that it does cover congruence of triangles. And that is what we are going to do in the next three lessons.

Axioms of Congruence

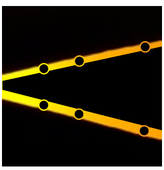
Points, lines, segments, rays, angles, triangles– we are starting to pile up a lot of objects here. At some point you are probably going to want to compare them to each other. You might have two different triangles in different locations, different orientations, but they have essentially the same shape, so you want to say that for practical purposes, they are equivalent. Well, congruence is a way to do that. Congruence, if you recall, is one of the undefined terms in Hilbert's system. Initially it describes a relation between a pair of segments or a pair of angles, so that we can say, for instance, that two segments are or are not congruent, or that two angles are or are not congruent. Later, the term is extended so that we can talk about congruence of triangles and other more general shapes. The notation used to indicate that two things (segments, angles, whatever) are congruent is $\simeq$. In Hilbert's system, there are six axioms of congruence. Three deal with congruence of segments, two deal with congruence of angles, and one involves both segments and angles.

The first and fourth of these make it possible to construct congruent copies of segments and angles wherever we want. They are a little reminiscent of Euclid's postulates in that way. The second and fifth axioms tell us that congruence is an equivalence relation. The third and sixth– well, I suppose that in a way they form a pair too– both deal with three points and the segments that have them as their endpoints. In the third axiom, the points are colinear, while in the sixth they are not. There is a more direct counterpart to the third axiom though, a statement which does for angles what the Segment Addition Axiom does for segments. It is called the Angle Addition Theorem and we will prove it in lesson 5.



I use a variety of symbols to mark segment and angle congruence.

Any time you throw something new into the mix, you probably want to figure out how it intermingles with what has come before. How does the new fit with the old? I realize that is a pretty vague question, but a more precise statement really depends upon the context. In our current situation, we have just added congruence to a system that already had incidence and order. The axioms of congruence themselves provide some basic connections between congruence and incidence and order. I think the most important remaining connection between congruence, incidence, and order is the Triangle Inequality, but that result is still a little ways away. In the meantime, the next theorem provides one more connection.



CONGRUENCE AND ORDER

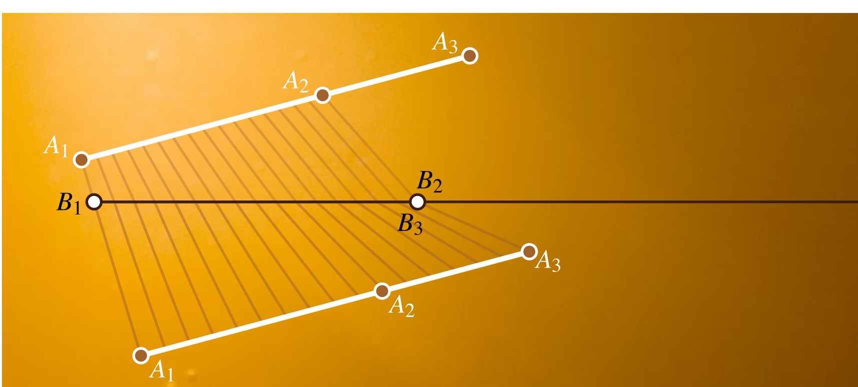
Suppose that $A_1 * A_2 * A_3$ and that B_3 is a point on the ray $B_1B_2\rightarrow$. If $A_1A_2 \simeq B_1B_2$ and $A_1A_3 \simeq B_1B_3$, then $B_1 * B_2 * B_3$.

Proof. Since B_3 is on $B_1B_2\rightarrow$ one of three things is going to happen:

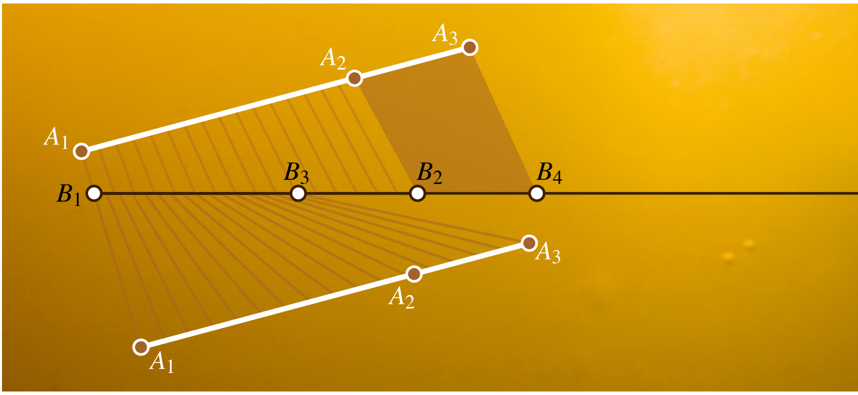
$$(1)\ B_2 = B_3 \quad (2)\ B_1 * B_3 * B_2 \quad (3)\ B_1 * B_2 * B_3.$$

The last is what we want, so it is just a matter of ruling out the other two possibilities.

(1) Why can't B_3 be equal to B_2? With $B_2 = B_3$, both A_1A_2 and A_1A_3 are congruent to the same segment. Therefore they are two different constructions of a segment starting from A_1 along $A_1A_2\rightarrow$ and congruent to B_1B_2. The Segment Construction Axiom says that there be only one.



The case against case I



The case against case II

(2) Why can't B_3 be between B_1 and B_2? By the Segment Construction Axiom, there is a point B_4 on the opposite side of B_2 from B_1 so that $B_2B_4 \simeq A_2A_3$. Now look:

$$B_1B_2 \simeq A_1A_2 \quad \& \quad B_2B_4 \simeq A_2A_3$$

so by the Segment Addition Axiom, $B_1B_4 \simeq A_1A_3$. This creates the same problem we ran into last time– two different segments B_1B_3 and B_1B_4, both starting from B_1 and going out along the same ray, yet both are supposed to be congruent to A_1A_3. □

Triangle Congruence

Congruence of segments and angles is undefined, subject only to the axioms of congruence. But congruence of triangles *is* defined. It is defined in terms of the congruences of the segments and angles that make up the triangles.

DEF: TRIANGLE CONGRUENCE
Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent if *all* of their corresponding sides and angles are congruent:

$$\begin{array}{lll} AB \simeq A'B' & BC \simeq B'C' & CA \simeq C'A' \\ \angle A \simeq \angle A' & \angle B \simeq \angle B' & \angle C \simeq \angle C'. \end{array}$$

Now that definition suggests that you have to match up six different things to say that two triangles are congruent. In actuality, triangles aren't really that flexible. Usually you only have to match up about half that many things. For example, the next result we will prove, the S·A·S Triangle Congruence Theorem, says that you only have to match up two sides of the triangles, and the angles between those sides, to show that the triangles are congruent. In this lesson, we begin the investigation of those minimum conditions.

Before we start studying these results, I would like to point out another way to view these theorems, this time in terms of *construction*. The triangle congruence theorems are set up to compare two triangles. Another way to think of them, though, is as a restriction on the way that a single triangle can be formed. To take an example, the S·A·S theorem below says that, modulo congruence, there is really only one triangle with a given pair of sides and a given intervening angle. Therefore, if you are building a triangle, and have decided upon two sides and an intervening angle, well, the triangle is decided– you don't get to choose the remaining side or the other two angles.

S·A·S TRIANGLE CONGRUENCE

In triangles $\triangle ABC$ and $\triangle A'B'C'$, if

$$AB \simeq A'B' \quad \angle B \simeq \angle B' \quad BC \simeq B'C',$$

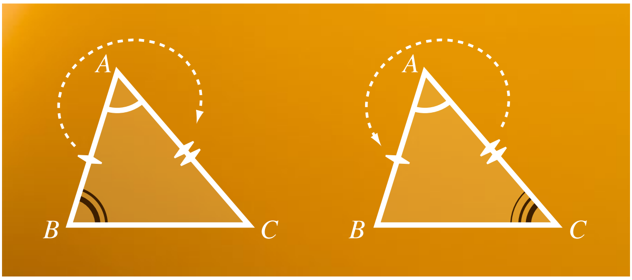
then $\triangle ABC \simeq \triangle A'B'C'$.

Proof. To show that two triangles are congruent, you have to show that three pairs of sides and three pairs of angles are congruent. Fortunately, two of the side congruences are given, and one of the angle congruences is given. The S·A·S axiom guarantees a second angle congruence, $\angle A \simeq \angle A'$. So that just leaves one angle congruence and one side congruence.

Let's do the angle first. You know, working abstractly creates a lot of challenges. On the few occasions when the abstraction makes things easier, it is a good idea to take advantage of it. This is one of those times. The S·A·S lemma tells us about $\angle A$ in $\triangle ABC$. But let's not be misled by lettering. Because $\triangle ABC = \triangle CBA$ and $\triangle A'B'C' = \triangle C'B'A'$, we can reorder the given congruences:

$$CB \simeq C'B' \quad \angle B \simeq \angle B' \quad BA \simeq B'A'.$$

Then the S·A·S lemma says that $\angle C \simeq \angle C'$. Sneaky isn't it? It is a completely legitimate use of the S·A·S axiom though.

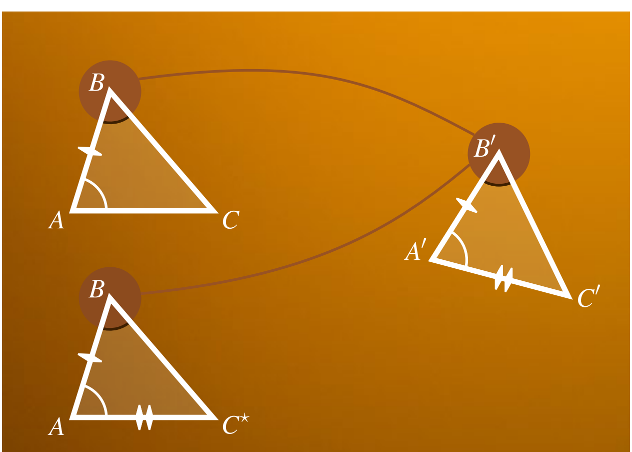


Two orderings of the list of congruences for the SAS lemma.

That just leaves the sides AC and $A'C'$. We are going to construct a congruent copy of $\triangle A'B'C'$ on top of $\triangle ABC$ (Euclid's flawed proof of S·A·S in *The Elements* used a similar argument but without the axioms to back it up). Thanks to the Segment Construction Axiom, there is a unique point $C^\star$ on AC so that $AC^\star \simeq A'C'$. Now if we can just show that $C^\star = C$ we will be done. Look:

$$BA \simeq B'A' \quad \angle A \simeq \angle A' \quad AC^\star \simeq \angle A'C'.$$

By the S·A·S axiom then, $\angle ABC^\star \simeq \angle A'B'C'$. That in turn means that $\angle ABC^\star \simeq \angle ABC$. But wait– both of those angles are constructed on the same side of $BA \rightarrow$. According to the Angle Construction Axiom, that means they must be the same. That is, $BC \rightarrow = BC^\star \rightarrow$. Both C and $C^\star$ are the intersection of this ray and the line AC. Since a ray can only intersect a line once, C and $C^\star$ do have to be the same. $\square$



To show the last sides are congruent, construct a third triangle from parts of the original two. The key to the location of C is the angle at B.

One of the things that I really appreciate about the triangle congruence theorems is how transparent they are: their names tell us when to use them. For instance, you use S·A·S when you know congruences for two sides and the angle between them. And you use A·S·A when...

A·S·A TRIANGLE CONGRUENCE

In triangles $\triangle ABC$ and $\triangle A'B'C'$, if

$$\angle A \simeq \angle A' \quad AB \simeq A'B' \quad \angle B \simeq \angle B',$$

then $\triangle ABC \simeq \triangle A'B'C'$.

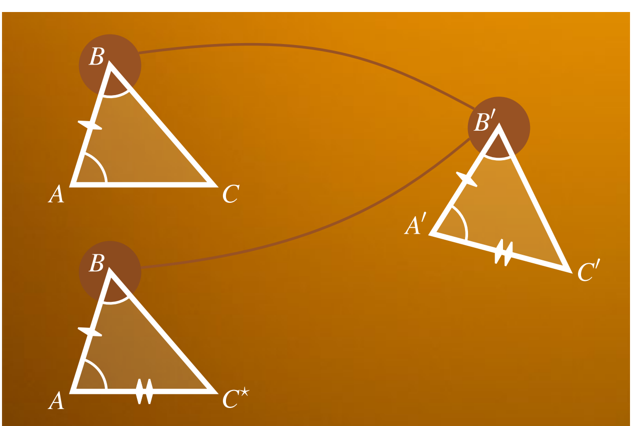
Proof. This time, it is a little easier–if we can just get one more side congruence, then S·A·S will provide the rest. You will probably notice some similarities between this argument and the last part of the S·A·S proof. Because of the Segment Construction Axiom, there is a point $C^\star$ on $AC\rightarrow$ so that $AC^\star \simeq A'C'$. Of course, the hope is that $C^\star = C$, and that is what we need to show. To do that, observe that

$$BA \simeq B'A' \quad \angle A \simeq \angle A' \quad AC^\star \simeq \angle A'C'.$$

By S·A·S, $\triangle ABC^\star \simeq \triangle A'B'C'$. In particular, look at what is happening at vertex B:

$$\angle ABC^\star \simeq \angle A'B'C' \simeq \angle ABC.$$

There is only one way to make that angle on that side of $BA\rightarrow$, and that means $BC^\star \rightarrow = BC\rightarrow$. Since both C and $C^\star$ are where this ray intersects $\leftarrow AC \rightarrow$, $C = C^\star$.



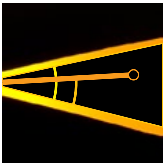
Does this look familiar?

That's the hard work. All that is left is to wrap up the argument. Since $C = C^\star$, $AC = AC^\star$, and that means $AC \simeq A'C'$. Then

$$BA \simeq B'A' \quad \angle A \simeq \angle A' \quad AC \simeq A'C'$$

so by S·A·S, $\triangle ABC \simeq \triangle A'B'C'$. □

Let's take a look at how the triangle congruence theorems can be put to work. This next theorem is the angle equivalent of the theorem at the start of this lesson relating congruence and the order of points.



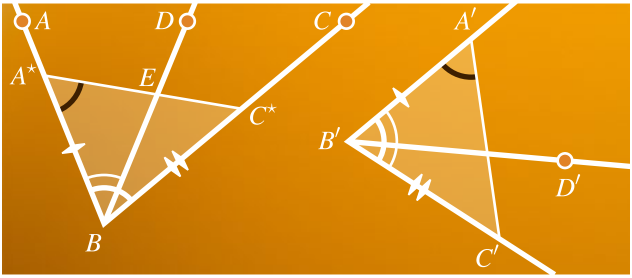
THM: CONGRUENCE AND ANGLE INTERIORS

Suppose that $\angle ABC \simeq \angle A'B'C'$. Suppose that D is in the interior of $\angle ABC$. And suppose that D' is located on the same side of $\leftarrow AB \rightarrow$ as C so that $\angle ABD \simeq \angle A'B'D'$. Then D' is in the interior of $\angle A'B'C'$.

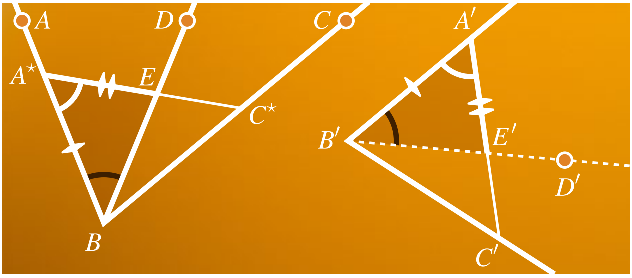
Proof. Because there is some flexibility in which points you choose to represent an angle, there is a good chance that our points are not organized in a very useful way. While we can't change the rays or the angles themselves, we can choose other points to represent them. So the first step is to reposition our points in the most convenient way possible. Let $A^\star$ be the point on $BA\rightarrow$ so that $BA^\star \simeq B'A'$. Let $C^\star$ be the point on $BC\rightarrow$ so that $BC^\star \simeq B'C'$. Since D is in the interior of $\angle ABC$, the Crossbar Theorem guarantees that $BD\rightarrow$ intersects $A^\star C^\star$. Let's call this intersection E. Then

$$A^\star B \simeq A'B' \quad \angle A^\star BC^\star \simeq A'B'C' \quad BC^\star \simeq B'C'$$

so by S·A·S, $\triangle A^\star BC^\star \simeq \triangle A'B'C'$.



After repositioning points, the first use of SAS.



The second use of SAS: E' and D' are on the same ray.

Okay, now let's turn our attention to the second configuration of points–the ones with the $'$ marks. According to the Segment Construction Axiom, there is a point E' on $A'C'\!\rightarrow$ so that $A'E' \simeq A^\star E$. Furthermore, thanks to the earlier theorem relating congruence and order, since E is between $A^\star$ and $C^\star$, E' must be between A' and C', and so it is in the interior of $\angle A'B'C'$. Now look:

$$BA^\star \simeq B'A' \quad \angle BA^\star E \simeq \angle B'A'E' \quad A^\star E \simeq A'E'$$

so by S·A·S, $\triangle BA^\star E \simeq \triangle B'A'E'$.

In particular, this means that $\angle A^\star BE \simeq \angle A'B'E'$. But we were originally told that $\angle A^\star BE \simeq \angle A'B'D'$. Since angle congruence is transitive this must mean that $\angle A'B'D' \simeq \angle A'B'E'$. Well, thanks to the Angle Construction Axiom, this means that the two rays $B'D'\!\rightarrow$ and $B'E'\!\rightarrow$ must be the same. Since E' is in the interior of $\angle A'B'C'$, D' must be as well. □

Symmetry in Triangles

I don't think it comes as a great surprise that in some triangles, two or even all three sides or angles may be congruent. Thanks to the triangle congruence theorems, we can show that these triangles are congruent to themselves in non-trivial ways. These non-trivial congruences reveal the internal symmetries of those triangles.

DEF: ISOSCELES, EQUILATERAL, SCALENE

If all three sides of a triangle are congruent, the triangle is *equilateral*. If exactly two sides of a triangle are congruent, the triangle is *isosceles*. If no pair of sides of the triangle is congruent, the triangle is *scalene*.

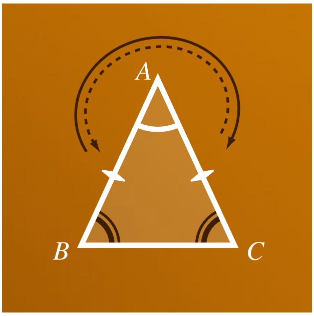
Here is one of those internal symmetry results. I put the others in the exercises.

THE ISOSCELES TRIANGLE THEOREM
In an isosceles triangle, the angles opposite the congruent sides are congruent.

Proof. Suppose $\triangle ABC$ is isosceles, with $AB \simeq AC$. Then

$$AB \simeq AC \quad \angle A \simeq \angle A \quad AC \simeq AB,$$

so by S·A·S, $\triangle ABC \simeq \triangle ACB$ (there's the non-trivial congruence of the triangle with itself). Comparing corresponding angles, $\angle B \simeq \angle C$. $\square$



Two orderings of the list of congruences for the SAS lemma.

Exercises

1. Given any point P and any segment AB, prove that there are infinitely many points Q so that $PQ \simeq AB$.

2. Verify that *triangle* congruence is an equivalence relation– that it is reflexive, symmetric, and transitive.

3. Prove the converse of the Isosceles Triangle Theorem: that if two interior angles of a triangle are congruent, then the sides opposite them must also be congruent.

4. Prove that all three interior angles of an equilateral triangle are congruent.

5. Prove that no two interior angles of a scalene triangle can be congruent.

6. In the exercises in Lesson 1, I introduced the Cartesian model and described how *point*, *line*, *on* and *between* are interpreted in that model. Let me extend that model now to include congruence. In the Cartesian model, segment congruence is defined in terms of the length of the segment, which, in turn, is defined using the distance function. If (x_a, y_a) and (x_b, y_b) are the coordinates of A and B, then the length of the segment AB, written $|AB|$, is

$$|AB| = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}.$$

   Two segments are congruent if and only if they are the same length. With this intepretation, verify the first three axioms of congruence.

7. Angle congruence is the most difficult to interpret in the Cartesian model. Like segment congruence, angle congruence is defined via measure– in this case angle measure. You may remember from calculus that the dot product provides a way to measure the angle between two vectors: that for any two vectors v and w,

$$v \cdot w = |v||w| \cos\theta,$$

   where θ is the angle between v and w. That is the key here. Given an angle $\angle ABC$, its measure, written $(\angle ABC)$, is computed as follows.

Let (x_a, y_a), (x_b, y_b) and (x_c, y_c) be the coordinates for points A, B, and C, then define vectors

$$v = \langle x_a - x_b, y_a - y_b \rangle \quad w = \langle x_c - x_b, y_c - y_b \rangle.$$

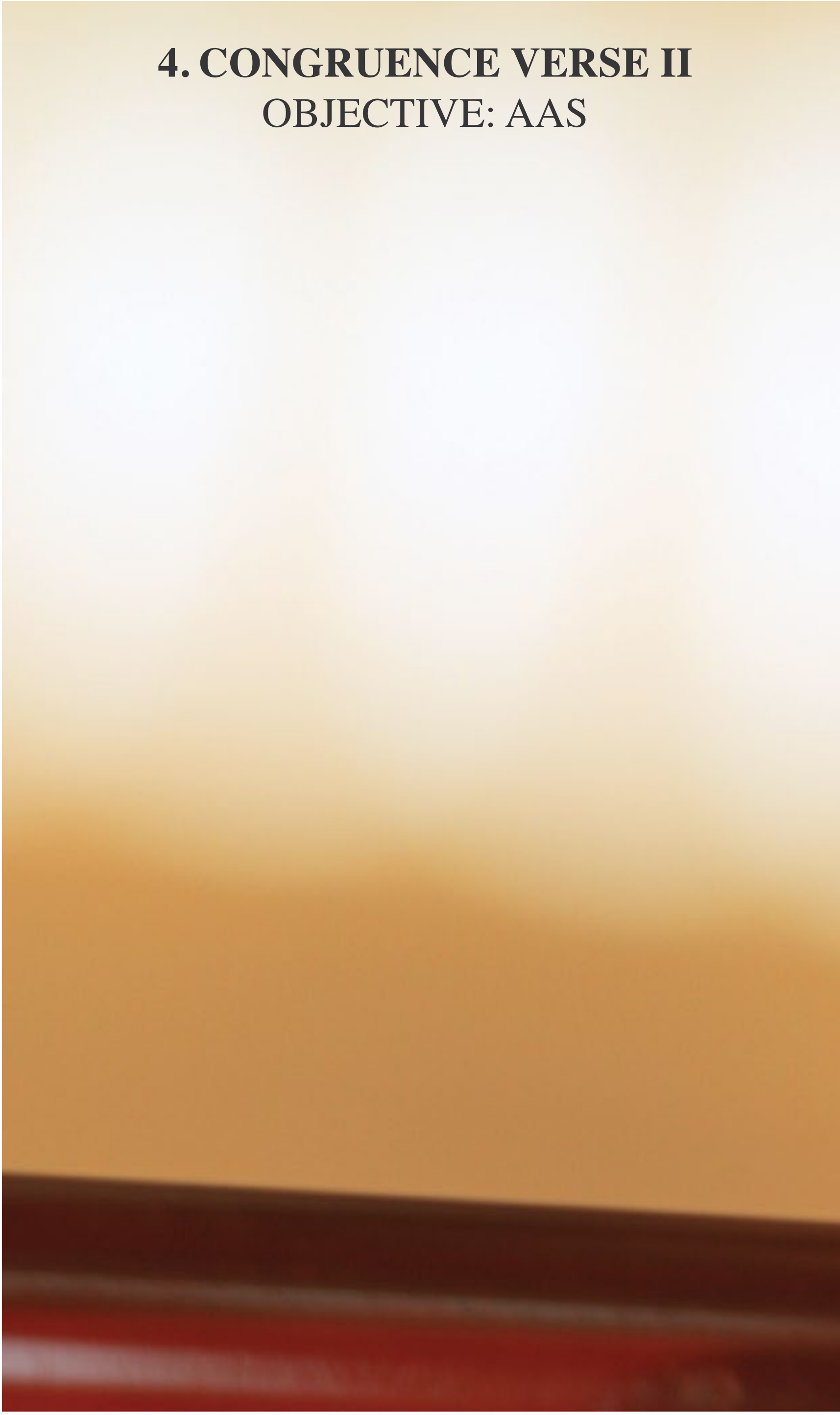
and measure

$$(\angle ABC) = \cos^{-1}\left(\frac{v \cdot w}{|v||w|}\right).$$

Two angles are congruent if and only if they have the same angle measure. With this interpretation, verify the last three axioms of congruence.

4. CONGRUENCE VERSE II

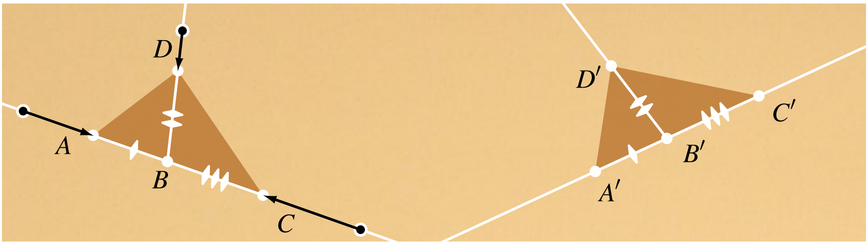
OBJECTIVE: AAS



The ultimate objective of this lesson is derive a third triangle congruence theorem, A·A·S. The basic technique I used in the last chapter to prove S·A·S and A·S·A does not quite work this time though, so along the way we are going to get to see a few more of the tools of neutral geometry: supplementary angles, the Alternate Interior Angle Theorem, and the Exterior Angle Theorem.

Supplementary Angles

There aren't that many letters in the alphabet, so it is easy to burn through most of them in a single proof if you aren't frugal. Even if your variables don't run the full gamut from A to Z, it can be a little challenging just trying to keep up with them. Some of this notation just can't be avoided; fortunately, some of it can. One technique I like to use to cut down on some notation is what I call "relocation". Let's say you are working with a ray $AB\rightarrow$. Now you can't change the endpoint A without changing the ray itself, but there is a little flexibility with the point B. If B' is any other point on the ray (other than A), then $AB\rightarrow$ and $AB'\rightarrow$ are actually the same. So rather than introduce a whole new point on the ray, I like to just "relocate" B to a more convenient location. The same kind of technique can also be used for angles and lines. Let me warn you: you must be careful not to abuse this relocation power. I have seen students relocate a point to one intersection, use the fact that the point is at that intersection in their proof, and then relocate it again a few steps later to another location. That is obviously bad! Yes there is some flexibility to the placement of some of these points, but once you have used up that flexibility, the point has to stay put.

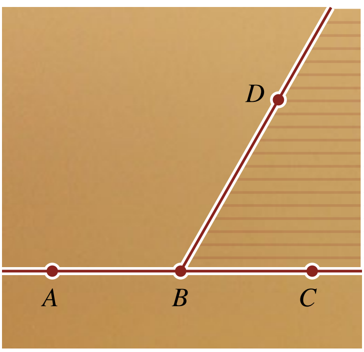


Relocation of points is a shortcut to cut down on notation. Illustrated here are the relocations of points A, B, and C to make the congruences needed for the proof that the supplements of congruent angles are congruent.

Three noncolinear points A, B, and C define an angle $\angle ABC$. When they are colinear, they do not define a proper angle, but you may want to think of them as forming a kind of degenerate angle. If $A * B * C$, then A, B, and C form what is called a "straight angle". One of the most basic relationships that two angles can have is defined in terms of these straight angles.

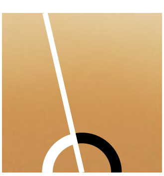
DEF: SUPPLEMENTARY ANGLES

Suppose that A, B and C form a straight angle with $A * B * C$. Let D be a fourth point which is not the line through A, B and C. Then $\angle ABD$ and $\angle CBD$ are *supplementary angles*.



A pair of supplementary angles: $\angle ABD$ and $\angle DBC$

Supplements have a nice and healthy relationship with congruence as related in the next theorem.



THM: CONGRUENT SUPPLEMENTS

The supplements of congruent angles are congruent: given two pairs of supplementary angles

Pair 1: $\angle ABD$ and $\angle CBD$ and

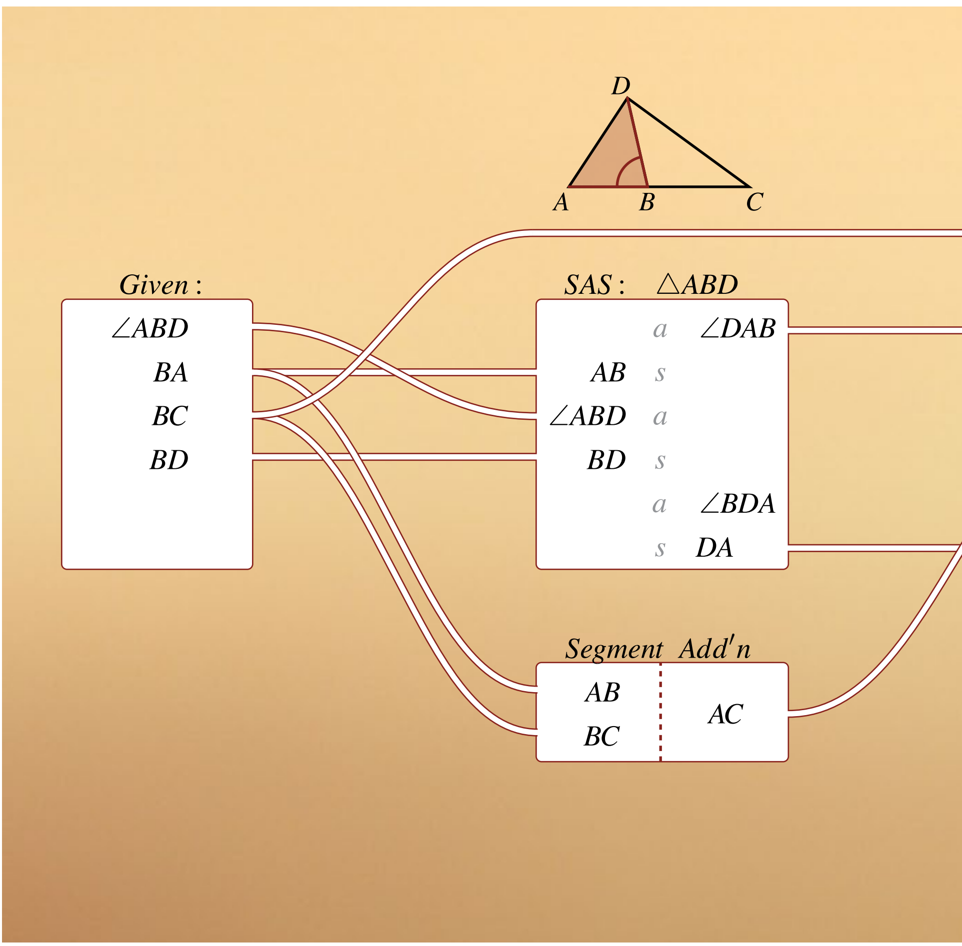
Pair 2: $\angle A'B'D'$ and $\angle C'B'D'$,

if $\angle ABD \simeq \angle A'B'D'$, then $\angle CBD \simeq \angle C'B'D'$.

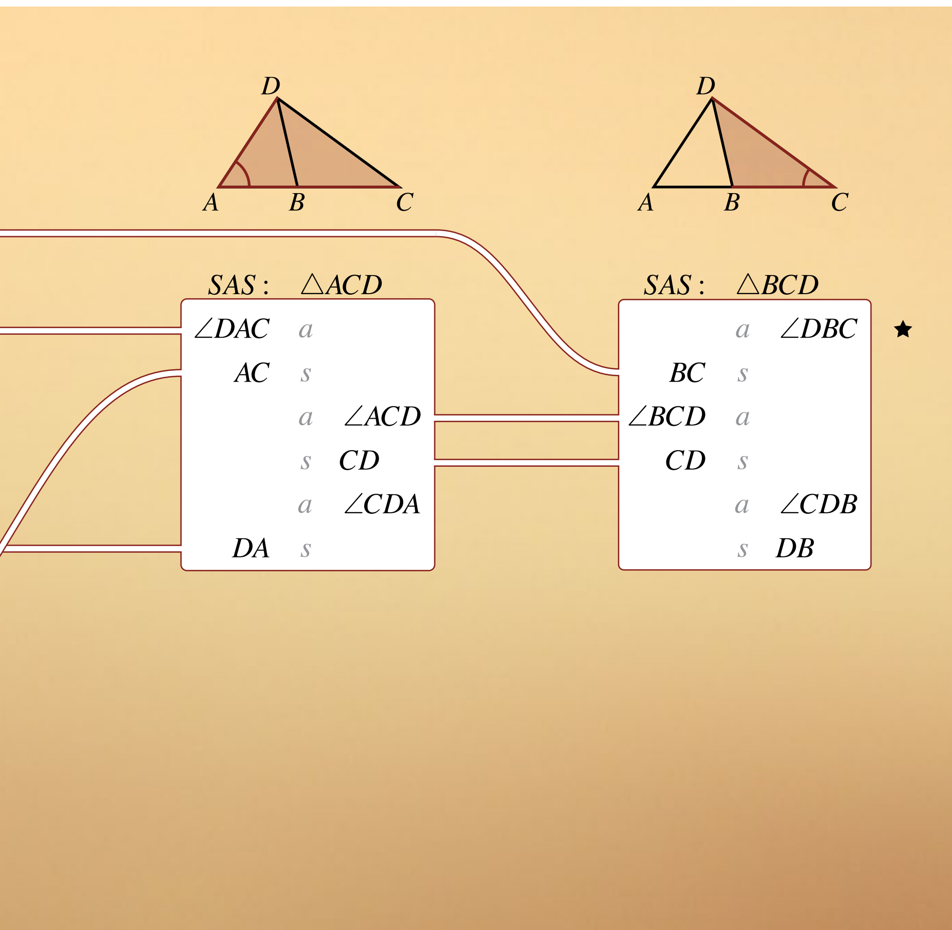
Proof. The idea is to relocate points to create a set of congruent triangles, and then to find a path of congruences leading from the given angles to the desired angle. In this case the relocation is easy enough: position A, C, and D on their respective rays $BA\rightarrow$, $BC\rightarrow$ and $BD\rightarrow$ so that

$$BA \simeq B'A' \quad BC \simeq B'C' \quad BD \simeq B'D'.$$

The path through the series of congruent triangles isn't that hard either if you just sit down to figure it out yourself. The problem is in writing it down so that a reader can follow along. In place of a traditional proof, I have made a chart that I think makes it easy to walk through the congruences. To read the chart, you need to know that I am using a little shorthand notation for each of the congruences. Here's the thing– each congruence throughout the entire proof compares segments, angles, or triangles with the *same* letters. The difference is that on the right hand side, the letters are marked with a $'$, while on the left they are not. For instance, the goal of this proof is to show that $\angle CBD \simeq \angle C'B'D'$. When I was working through the proof I found it a little tedious have to write the whole congruence out with every single step. Since the left hand side of

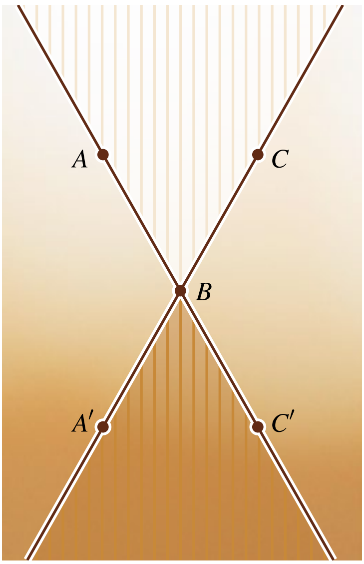


the congruence determines the right hand side anyway, I just got in the habit of writing down only the left hand side. In the end I decided that was actually easier to read than the whole congruence, so in the chart, the statement AB really means $AB \simeq A'B'$. I still feel a little uneasy doing this, so let me give another defense of this shorthand. One of the things I talked about in the last lesson was the idea of these congruences "locking in" a triangle– if you know S·A·S, for instance, then the triangle is completely determined. The statements in this proof can be interpreted as the locking in of various segments, angles, and triangles. For instance, B is between A and C, so if AB and BC are given, then AC is locked in by the Segment Addition Axiom. Okay, so that's enough about the notation. Here's the chart of the proof.



Every angle has two supplements. To get a supplement of an angle, simply replace one of the two rays forming the angle with its opposite ray. Since there are two candidates for this replacement, there are two supplements. There is a name for the relationship between these two supplements.

DEF: VERTICAL ANGLES
Vertical angles are two angles which are supplementary to the same angle.



Two intersecting lines generate two pairs of vertical angles.

Pair 1: $\angle ABC$ *and* $\angle A'BC'$
Pair 2: $\angle ABA'$ *and* $\angle CBC'$

Every angle is part of one and only one vertical angle pair (something you may want to prove). For $\angle ABC$, the other half of the pair is the angle formed by the rays $(BA\rightarrow)^{op}$ and $(BC\rightarrow)^{op}$. Without a doubt, the single most important property of vertical angles is that

THM: ON VERTICAL ANGLES
Vertical angles are congruent.

Proof. Two vertical angles are, by definition, supplementary to the same angle. That angle is congruent to itself (because of the second axiom of congruence). Now we can use the last theorem. Since the vertical angles are supplementary to congruent angles, they themselves must be congruent. □

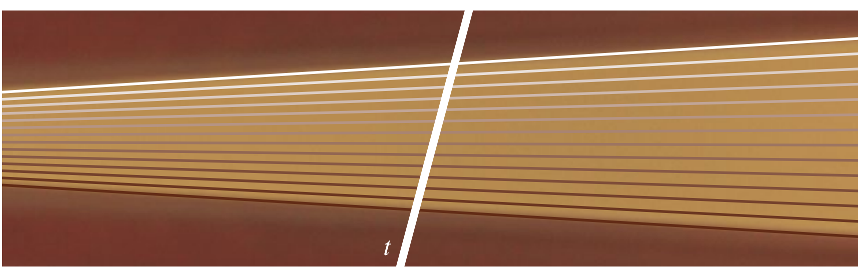
The Alternate Interior Angle Theorem

The farther we go in the study of neutral geometry, the more we are going to bump into issues relating to how parallel lines behave. A lot of the results we will derive are maddeningly close to results of Euclidean geometry, and this can lead to several dangerous pitfalls. The Alternate Interior Angle Theorem is maybe the first glimpse of that.

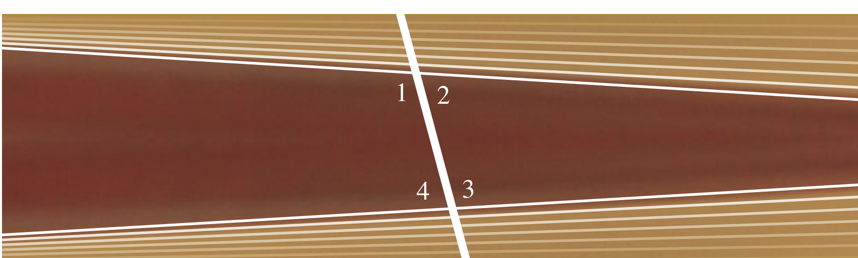
DEF: TRANSVERSALS
Given a set of lines, $\{\ell_1, \ell_2, \ldots, \ell_n\}$, a *transversal* is a line which intersects all of them.

DEF: ALTERNATE AND ADJACENT INTERIOR ANGLES
Let t be a transversal to ℓ_1 and ℓ_2. *Alternate interior angles* are pairs of angles formed by ℓ_1, ℓ_2, and t, which are between ℓ_1 and ℓ_2, and on opposite sides of t. *Adjacent interior angles* are pairs of angles on the same side of t.

The Alternate Interior Angle Theorem tells us something about transversals and parallel lines. Read it carefully though. The converse of this theorem is used a lot in Euclidean geometry, but in neutral geometry this is *not* an "if and only if" statement.



A transversal t of a set of lines.



Alternate pairs: 1 *and* 3, 2 *and* 4. *Adjacent pairs:* 1 *and* 4, 2 *and* 3.

THE ALTERNATE INTERIOR ANGLE THEOREM
Let ℓ_1 and ℓ_2 be two lines, crossed by a transversal t. If the alternate interior angles formed are congruent, then ℓ_1 and ℓ_2 are parallel.

Proof. First I want to point out something that may not be entirely clear in the statement of the theorem. The lines ℓ_1, ℓ_2 and t will actually form *two* pairs of alternate interior angles. However, the angles in one pair are the supplements of the angles in the other pair, so if the angles in one pair are congruent then angles in the other pair also have to be congruent. Now let's get on with the proof, a proof by contradiction. Suppose that ℓ_1 and ℓ_2 are crossed by a transversal t so that alternate interior angles are congruent, but suppose that ℓ_1 and ℓ_2 are not parallel. Label

A: the intersection of ℓ_1 and t;

B: the intersection of ℓ_2 and t;

C: the intersection of ℓ_1 and ℓ_2.

By the Segment Construction Axiom there are also points

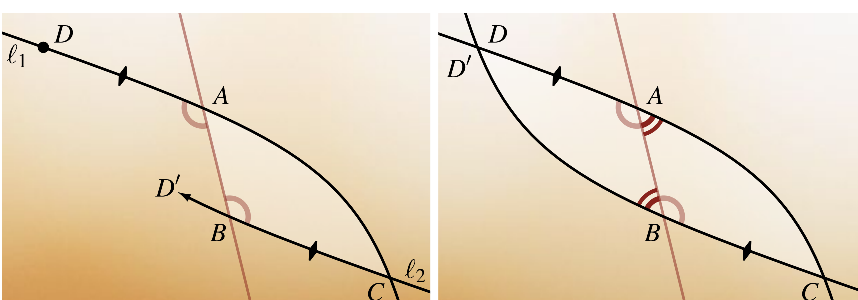
D on ℓ_1 so that $D * A * C$ and so that $AD \simeq BC$, and

D' on ℓ_2 so that $D' * B * C$ and so that $BD' \simeq AC$.

In terms of these marked points the congruent pairs of alternate interior angles are

$$\angle ABC \simeq \angle BAD \quad \& \quad \angle ABD' \simeq \angle BAC.$$

Take the first of those congruences, together with the fact that that we have constructed $AD \simeq BC$ and $AB \simeq BA$, and that's enough to use S·A·S:



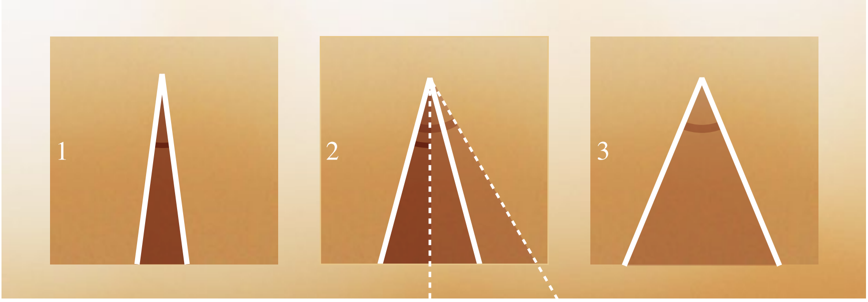
If ℓ_1 and ℓ_2 crossed on one side of t, they would have to cross on the other side.

$\triangle ABC \simeq \triangle BAD$. I really just want to focus on one pair of corresponding angles in those triangles though: $\angle ABD \simeq \angle BAC$. Now $\angle BAC$ is congruent to its alternate interior pair $\angle ABD'$, so since angle congruence is transitive, this means that $\angle ABD \simeq \angle ABD'$. Here's the problem. There is only one way to construct this angle on that side of t, so the rays $BD\rightarrow$ and $BD'\rightarrow$ must actually be the same. That means that D, which we originally placed on ℓ_1, is also on ℓ_2. That would imply that ℓ_1 and ℓ_2 share two points, C and D, in violation of the very first axiom of incidence. □

The Exterior Angle Theorem

We have talked about congruent angles, but so far we have not discussed any way of saying that one angle is larger or smaller than the other. That is something that we will need to do eventually, in order to develop a system of measurement for angles. For now though, we need at least some rudimentary definitions of this, even if the more fully developed system will wait until later.

DEF: SMALLER AND LARGER ANGLES
Given two angles $\angle A_1B_1C_1$ and $\angle A_2B_2C_2$, the Angle Construction Axiom guarantees that there is a point $A^\star$ on the same side of $\leftarrow B_2C_2\rightarrow$ as A_2 so that $\angle A^\star B_2C_2 \simeq \angle A_1B_1C_1$. If $A^\star$ is in the interior of $\angle A_2B_2C_2$, then we say that $\angle A_1B_1C_1$ is *smaller* than $\angle A_2B_2C_2$. If $A^\star$ is on the ray B_2C_2, then the two angles are congruent as we have previously seen. If $A^\star$ is neither in the interior of $\angle A_2B_2C_2$, nor on the ray $B_2C_2\rightarrow$, then $\angle A_1B_1C_1$ is *larger* than $\angle A_2B_2C_2$.

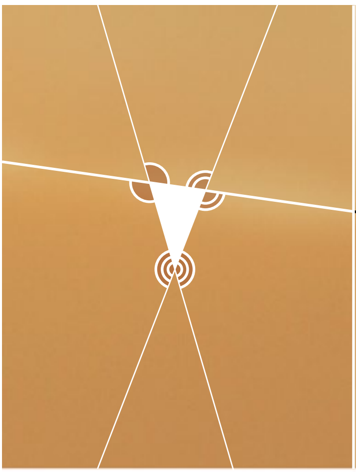


$\angle 1$ is smaller than $\angle 2$ $\angle 3$ is larger than $\angle 2$

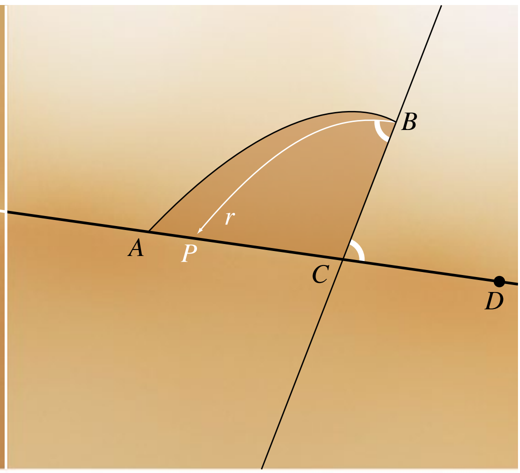
In lesson 8, I will come back to this in more detail. Feel free to skip ahead if you would like a more detailed investigation of this way of comparing non-congruent angles.

DEF: EXTERIOR ANGLES

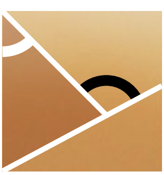
An *exterior angle* of a triangle is an angle supplementary to one of the triangle's interior angles.



Three pairs of exterior angles



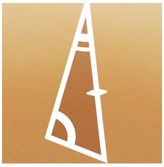
Exterior Angle Th'm: a proof by contradiction



THE EXTERIOR ANGLE THEOREM

The measure of an exterior angle of a triangle is greater than the measure of either of the nonadjacent interior angles.

Proof. I will use a straightforward proof by contradiction. Starting with the triangle $\triangle ABC$, extend the side AC past C: just pick a point D so that $A * C * D$. Now suppose that the interior angle at B is larger than the exterior angle at $\angle BCD$. Then there is a ray r from B on the same side of BC as A so that $BC \rightarrow$ and r form an angle congruent to $\angle BCD$. This ray will lie in the interior of $\angle B$, though, so by the Crossbar Theorem, r must intersect AC. Call this intersection point P. Now wait, though. The alternate interior angles $\angle PBC$ and $\angle BCD$ are congruent. According to the Alternate Interior Angle Theorem r and AC must be parallel– they can't intersect. This is an contradiction. $\square$



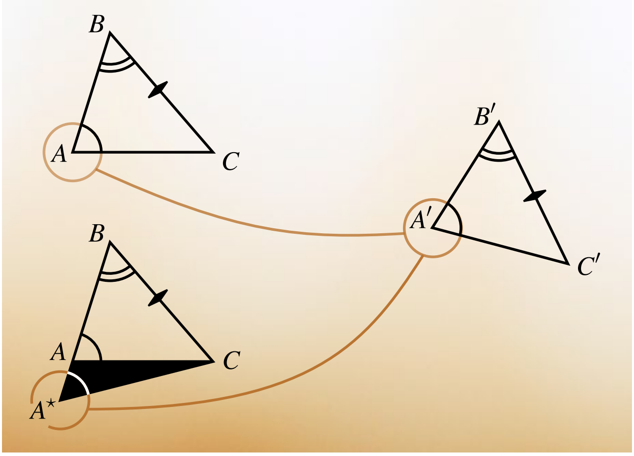
A·A·S TRIANGLE CONGRUENCE

In triangles $\triangle ABC$ and $\triangle A'B'C'$, if

$$\angle A \simeq \angle A' \quad \angle B \simeq \angle B' \quad BC \simeq B'C',$$

then $\triangle ABC \simeq \triangle A'B'C'$.

Proof. The setup of this proof is just like the proof of A·S·A, but for the critical step we are going to need to use the Exterior Angle Theorem. Locate $A^\star$ on $BA\rightarrow$ so that $A^\star B \simeq A'B'$. By S·A·S, $\triangle A^\star BC \simeq \triangle A'B'C'$. Therefore $\angle A^\star \simeq \angle A' \simeq \angle A$. Now if $B * A * A^\star$ (as illustrated) then $\angle A$ is an exterior angle and $\angle A^\star$ is a nonadjacent interior angle of the triangle $\triangle AA^\star C$. Acording to the Exterior Angle Theorem, these angles can't be congruent. If $B * A * A^\star$, then $\angle A^\star$ is an exterior angle and $\angle A$ is a nonadjacent interior angle. Again, the Exterior Angle Theorem says these angles can't be congruent. The only other possibility, then, is that $A = A^\star$, so $AB \simeq A'B'$, and by S·A·S, that means $\triangle ABC \simeq \triangle A'B'C'$. □



A familiar chase to prove AAS, but this time we have to call upon the Exterior Angle Theorem.

Exercises

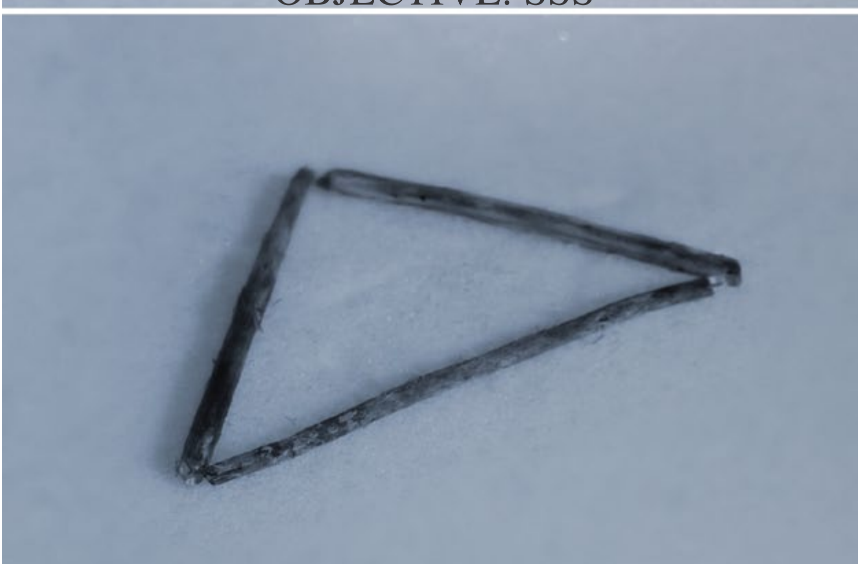
1. Prove that for every segment AB there is a point M on AB so that $AM \simeq MB$. This point is called the *midpoint* of AB.

2. Prove that for every angle $\angle ABC$ there is a ray $BD \rightarrow$ in the interior of $\angle ABC$ so that $\angle ABD \simeq \angle DBC$. This ray is called the *bisector* of $\angle ABC$.

3. Working from the spaghetti diagram proof that the supplements of congruent angles are congruent, write a traditional proof.

5. CONGRUENCE VERSE III

OBJECTIVE: SSS



In the last lesson I pointed out that the first and second axioms of congruence have angle counterparts in the fourth and fifth axioms, but that there was no direct angle counterpart to the third axiom, the Segment Addition Axiom. The next couple of results fill that hole.

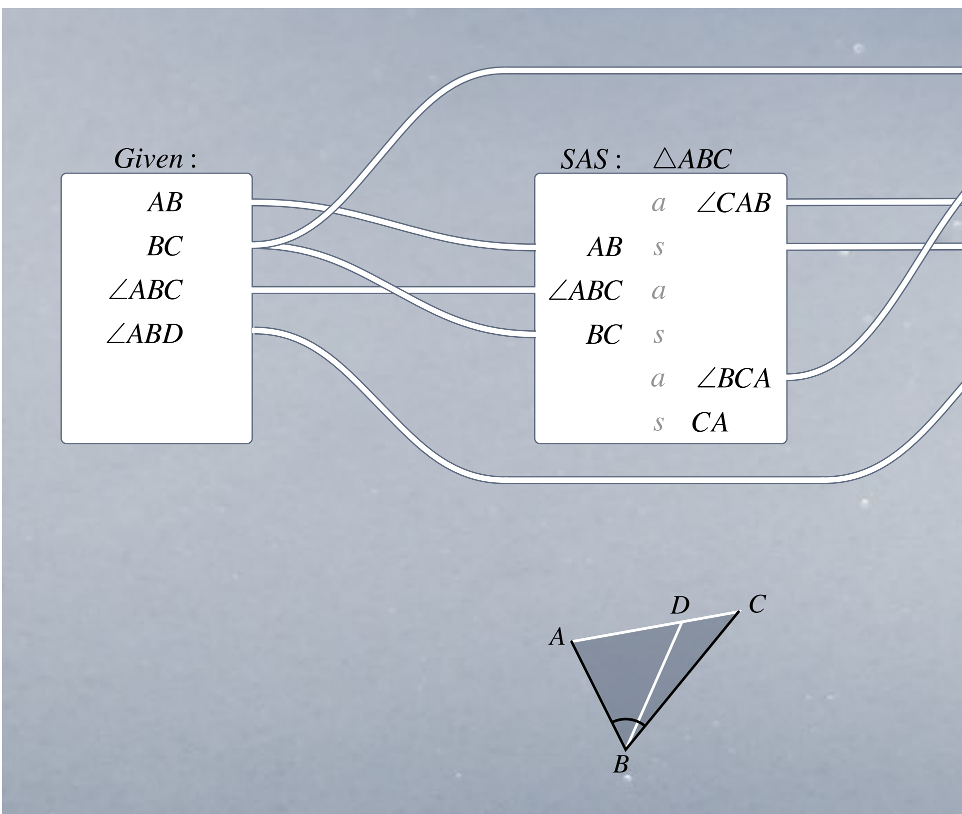
THE ANGLE SUBTRACTION THEOREM

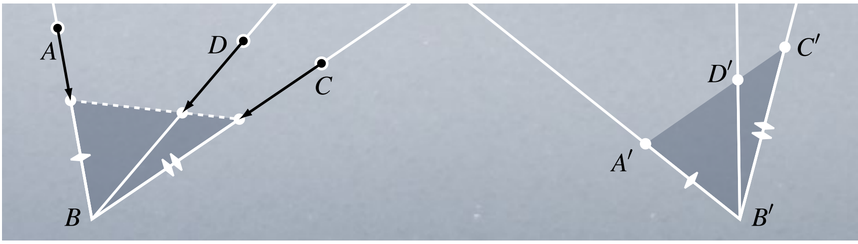
Let D and D' be interior points of $\angle ABC$ and $\angle A'B'C'$ respectively. If

$$\angle ABC \simeq \angle A'B'C' \quad \& \quad \angle ABD \simeq \angle A'B'D',$$

then $\angle DBC \simeq \angle D'B'C'$.

Proof. This proof is a lot like the proof that supplements of congruent angles are congruent, and I am going to take the same approach. The first step is one of relocation. Relocate A and C on $BA\rightarrow$ and $BC\rightarrow$ respectively

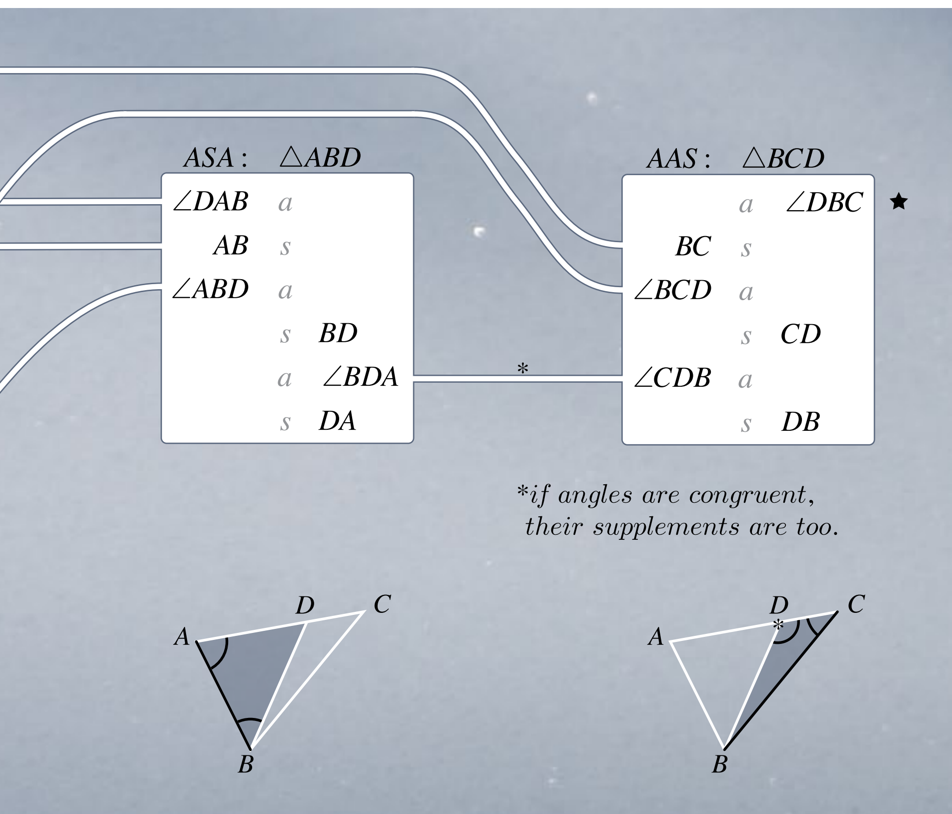




so that

$$BA \simeq B'A' \quad \& \quad BC \simeq B'C'.$$

Since D is in the interior of $\angle ABC$, by the Crossbar Theorem, $BD \rightarrow$ intersects AC. Relocate D to that intersection. Likewise, relocate D' to the intersection of $B'D' \rightarrow$ and $A'C'$. Note that this does not mean that $BD \simeq B'D'$ although that is something that we will establish in the course of the proof. I am going to use a chart to illustrate the congruences in place of a "formal" proof. □



With angle subtraction in the toolbox, angle addition is now easy to prove.

THE ANGLE ADDITION THEOREM

Suppose that D is in the interior of $\angle ABC$ and that D' is in the interior of $\angle A'B'C'$. If

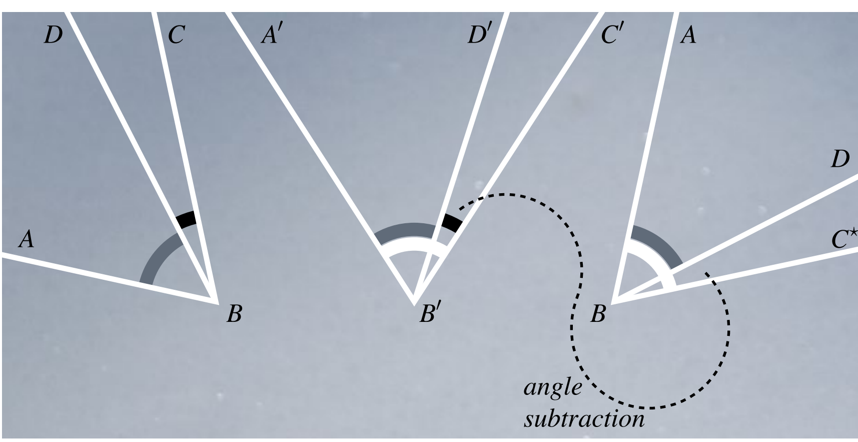
$$\angle ABD \simeq \angle A'B'D' \quad \& \quad \angle DBC \simeq \angle D'B'C',$$

then $\angle ABC \simeq \angle A'B'C'$.

Proof. Because of the Angle Construction Axiom, there is a ray $BC^{\star}\!\rightarrow$ on the same side of $\leftarrow AB \rightarrow$ as C so that $\angle ABC^{\star} \simeq \angle A'B'C'$. What we will show here is that $BC\!\rightarrow$ and $BC^{\star}\!\rightarrow$ are actually the same so that the angles $\angle ABC$ and $\angle ABC^{\star}$ are the same as well. This all boils down to one simple application of the Angle Subtraction Theorem:

$$\angle ABC^{\star} \simeq \angle A'B'C' \quad \& \quad \angle ABD \simeq \angle A'B'D' \quad \Longrightarrow \quad \angle DBC^{\star} \simeq \angle D'B'C'.$$

We already know that $\angle D'B'C' \simeq \angle DBC$, so $\angle DBC^{\star} \simeq \angle DBC$. The Angle Construction Axiom tells us that there is but one way to construct this angle on this side of $\leftarrow\! DB \!\rightarrow$, so $BC^{\star}\!\rightarrow$ and $BC\!\rightarrow$ have to be the same. $\square$



The proof of the Angle Addition Theorem.

We end this lesson with the last of the triangle congruence theorems. The proofs of the previous congruence theorems all used essentially the same approach, but that approach required an angle congruence. No angle congruence is given this time, so that won't work. Instead we are going to be using the Isosceles Triangle Theorem.

S · S · S TRIANGLE CONGRUENCE

In triangles $\triangle ABC$ and $\triangle A'B'C'$ if

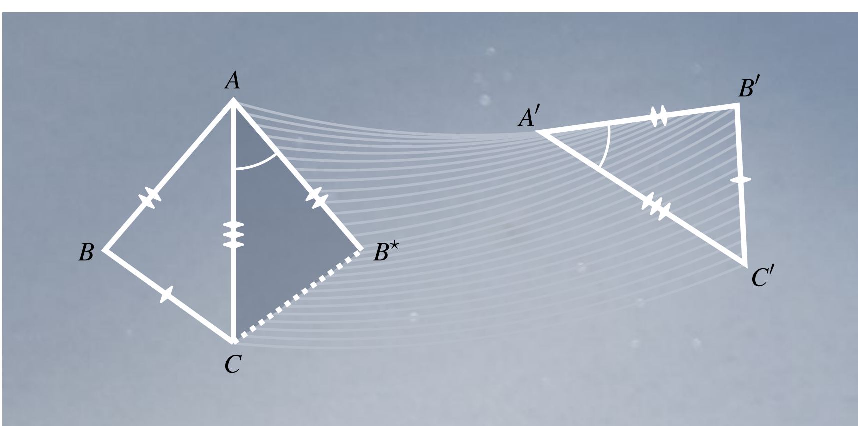
$$AB \simeq A'B' \quad BC \simeq B'C' \quad CA \simeq C'A',$$

then $\triangle ABC \simeq \triangle A'B'C'$.

Proof. The first step is to get the two triangles into a more convenient configuration. To do that, we are going to create a congruent copy of $\triangle A'B'C'$ on the opposite side of $\leftarrow AC \rightarrow$ from B. The construction is simple enough: there is a unique point $B^\star$ on the opposite side of $\leftarrow AC \rightarrow$ from B such that:

$$\angle CAB^\star \simeq \angle C'A'B' \quad \& \quad AB^\star \simeq A'B'.$$

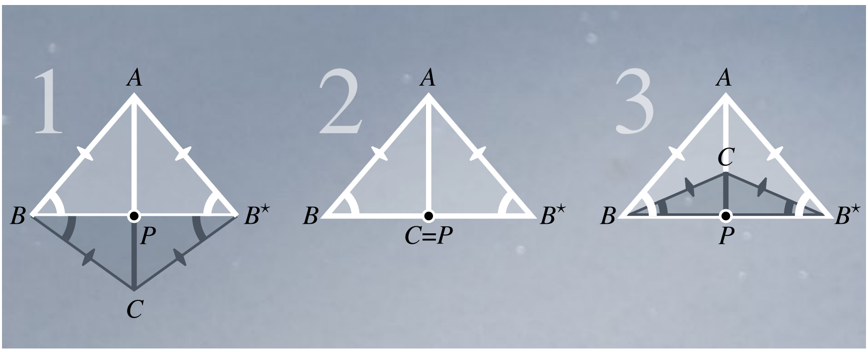
In addition, we already know that $AC \simeq A'C'$, so by S·A·S, $\triangle ABC^\star$ is congruent to $\triangle A'B'C'$. Now the real question is whether $\triangle ABC^\star$ is congruent to $\triangle ABC$, and that is the next task.



Creating a congruent copy of the second triangle abutting the first triangle.

Since B and $B^\star$ are on opposite sides of $\leftarrow AC \rightarrow$, the segment $BB^\star$ intersects $\leftarrow AC \rightarrow$. Let's call that point of intersection P. Now we don't know anything about where P is on $\leftarrow AC \rightarrow$, and that opens up some options:

(1) P could be between A and C, or
(2) P could be either of the endpoints A or C, or
(3) P could be on the line $\leftarrow AC \rightarrow$ but not the segment AC.



Three possible locations of P, and the resulting isosceles triangles.

I am just going to deal with that first possibility. If you want a complete proof, you are going to have to look into the remaining two cases yourself. Assuming that $A * P * C$, both of the triangles $\triangle ABB^\star$ and $\triangle CBB^\star$ are isosceles:

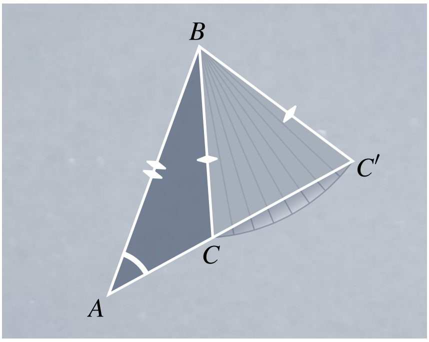
$$AB \simeq A'B' \simeq AB^\star$$
$$CB \simeq C'B' \simeq CB^\star.$$

According to the Isosceles Triangle Theorem, the angles opposite those congruent sides are themselves congruent:

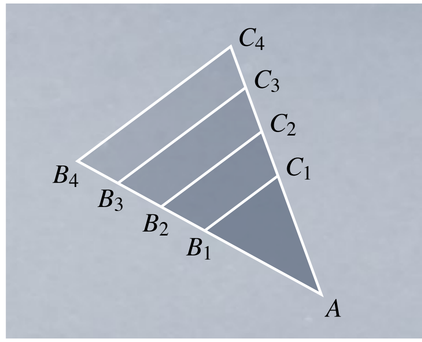
$$\angle ABP \simeq \angle AB^\star P$$
$$\angle CBP \simeq \angle CB^\star P.$$

Since we are assuming that P is between A and C, we can use the Angle Addition Theorem to combine these two angles into the larger angle $\angle ABC \simeq \angle AB^\star C$. We already know $\angle AB^\star C \simeq \angle A'B'C'$, so $\angle ABC \simeq \angle A'B'C'$ and that is the needed angle congruence. By S·A·S, $\triangle ABC \simeq \triangle A'B'C'$. □

Failures of S·S·A and A·A·A in the Euclidean model.



Two noncongruent triangles sharing S·S·A.



Four noncongruent triangles sharing A·A·A.

We have established four triangle congruences: S·A·S, A·S·A, A·A·S, and S·S·S. For each, you need three components, some mix of sides and angles. It would be natural to wonder whether there are any other combinations of three sides and angles which give a congruence. There are really only two other fundamentally different combinations: A·A·A and S·S·A. Neither is a valid congruence theorem in neutral geometry. In fact, both fail in Euclidean geometry. The situation in non-Euclidean geometry is a little bit different, but I am going to deflect that issue for the time being.

Exercises

1. The Segment Addition Axiom. The Angle Subtraction Theorem. The Angle Addition Theorem. That just leaves the Segment Subtraction Theorem. State it. Prove it.

2. One of the conditions in the statement of the Angle Subtraction Theorem is that both D and D' must be in the interiors of ther respective angles. In fact, this condition can be weakened: prove that you do not need to assume that D' is in the interior of the angle, just that it is on the same side of $A'B'$ as C'.

3. Complete the proof of S·S·S by handling the other two cases (when P is one of the endpoints and when P is on the line $\leftarrow AC \rightarrow$ but not the segment AC).

4. Suppose that $A * B * C$ and that A' and C' are on opposite sides of $\leftarrow AC \rightarrow$. Prove that if $\angle ABA' \simeq \angle CBC'$, then $A' * B * C'$.

5. Suppose that A, B, C, and D are four distinct non-colinear points. Prove that if $\triangle ABC \simeq \triangle DCB$, then $\triangle BAD \simeq \triangle CDA$.

6. READER’S SOLO
SHORTER AND LONGER

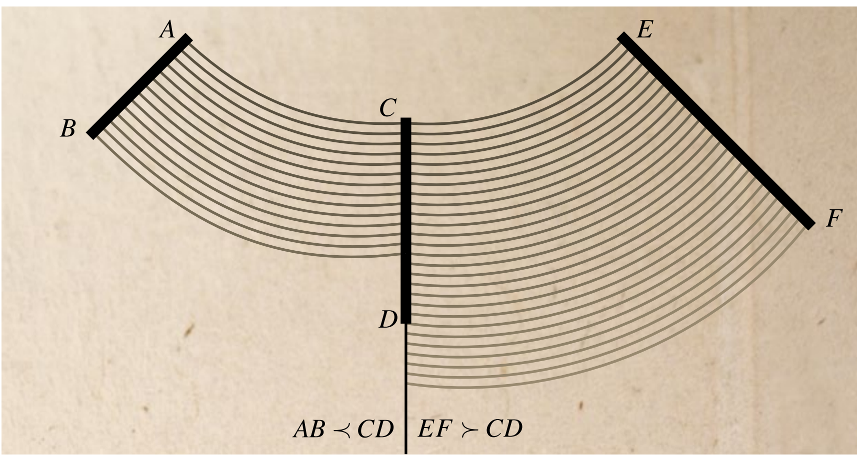
The purpose of this short section is to develop a system of comparison for segments that aren't congruent. I am going to let you provide all the proofs in this section. It will give you the opportunity to work with order and congruence on your own.

DEF: SHORTER AND LONGER

Given segments AB and CD, label E on $CD\rightarrow$ so that $CE \simeq AB$.

If $C * E * D$, then AB is *shorter than* CD, written $AB \prec CD$.

If $C * D * E$, then AB is *longer than* CD, written $AB \succ CD$.



Note that if you replace CD in this definition with DC, things will change slightly: calculations will be done on the ray $DC\rightarrow$ rather than $CD\rightarrow$. That would seem like it could be problem, since CD and DC are actually the same segment, so your first task in this chapter is to make sure that $\prec$ and $\succ$ are defined the same way, whether you are using CD or DC.

THM: $\prec$ AND $\succ$ ARE WELL DEFINED

Given segments AB and CD, label:

E: the unique point on $CD\rightarrow$ so that $AB \simeq CE$ and

F: the unique point on $DC\rightarrow$ so that $AB \simeq DF$.

Then $C * E * D$ if and only if $D * F * C$.

Here are a bunch of the properties of $\prec$ for you to verify. There are, of course, corresponding properties for $\succ$, but I have left them out to cut down on some of the tedium.

THM: TRANSITIVITY OF $\prec$

If $AB \prec CD$, and $CD \prec EF$, then $AB \prec EF$.
If $AB \prec CD$, and $CD \simeq EF$, then $AB \prec EF$.
If $AB \simeq CD$, and $CD \prec EF$, then $AB \prec EF$.

THM: SYMMETRY BETWEEN $\prec$ AND $\succ$

For any two segments AB and CD, $AB \prec CD$ if and only if $CD \succ AB$.

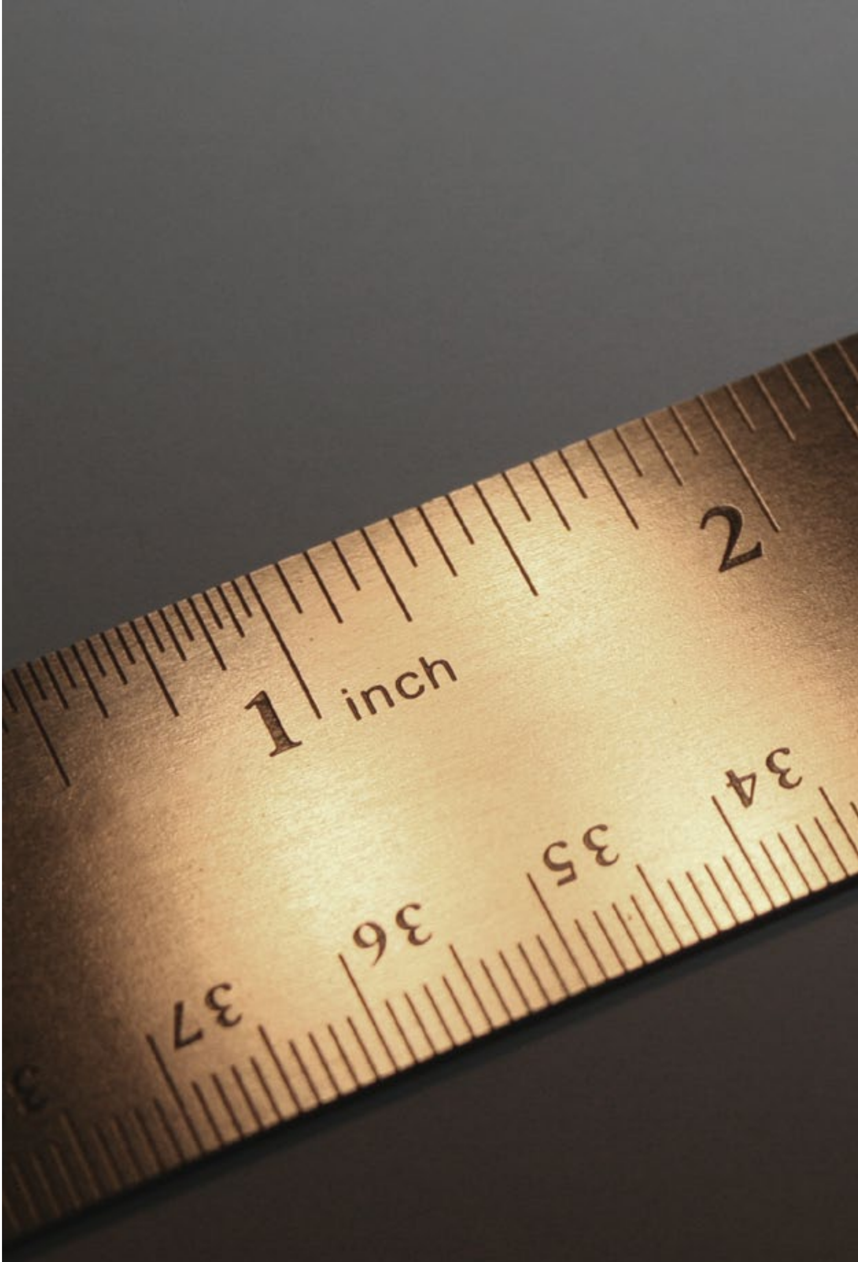
THM: ORDER (FOUR POINTS) AND $\prec$

If $A * B * C * D$, then $BC \prec AD$.

THM: ADDITIVITY OF $\prec$

Suppose that $A * B * C$ and $A' * B' * C'$. If $AB \prec A'B'$ and $BC \prec B'C'$, then $AC \prec A'C'$.

7. FILL THE HOLE
DISTANCE, LENGTH, AND THE AXIOMS OF CONTINUITY



Hilbert's geometry starts with incidence, congruence, and order. It is a synthetic geometry in the sense that it is not centrally built upon measurement. Nowadays, it is more common to take an metrical approach to geometry, and to establish your geometry based upon a measurement. In the metrical approach, you begin by defining a distance function– a function d which assigns to each pairs of points a real number and satisfies the following requirements

(i) $d(P,Q) \geq 0$, with $d(P,Q) = 0$ if and only if $P = Q$,
(ii) $d(P,Q) = d(Q,P)$, and
(iii) $d(P,R) \leq d(P,Q) + d(Q,R)$.

Once the distance function has been chosen, the length of a segment is defined to be the distance between its endpoints. I will follow the convention of using the absolute value sign to notate the length of a segment, so $|PQ| = d(P,Q)$. Then congruence is defined by saying that two segments are congruent if they have the same length. Incidence and order also can be defined in terms of d: points P, Q, and R are all on the same line, and Q is between P and R when the inequality in (iii) is an equality. You see, synthetic geometry takes a back seat to analytic geometry, and the synthetic notions of incidence, order, and congruence, are defined analytically. I do not have a problem with that approach– it is the one that we are going to take in the development of hyperbolic geometry much later on. We have been developing a synthetic geometry, though, and so what I would like to do in this lesson is to build distance out of incidence, order, and congruence. This is what Hilbert did when he developed the real number line and its properties inside of the framework of his axiomatic system.

Modest Expectations

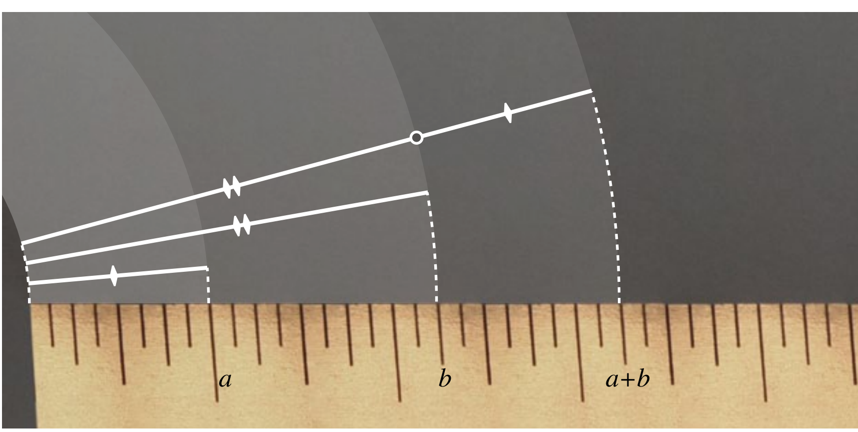
Here we stand with incidence, order, congruence, the axioms describing them, and at this point even a few theorems. Before we get out of this section, I will throw in the last two axioms of neutral geometry, the axioms of continuity, too. From all of this, we want to build a distance function d. Look, we have all dealt with distance before in one way or another, and we want our distance function to meet conditions (i)–(iii) above, so it is fair to have certain expectations for d. I don't think it is unreasonable to expect all of the following.

(1) The distance between any two distinct points should be a positive real number and the distance from a point to itself should be zero. That way, d will satisfy condition (i) above.

(2) Congruent segments should have the same length. That takes care of condition (ii) above, since $AB \simeq BA$, but it does a whole lot more too. You see, let's pick out some ray r and label its endpoint O. According to the Segment Construction Axiom, for any segment AB, there is a unique point P on r so that $AB \simeq OP$. If congruent segments are to have the same length, then that means $|AB| = d(O,P)$. Therefore, if we can just work out the distance from O to the other points on r, then all other distances will follow.

(3) If $A * B * C$, then
$$|AB| + |BC| = |AC|.$$
This is just a part of property (iii) of a distance function. Since we are going to develop the distance function on r, we don't have to worry about non-colinear points just yet (that will come a little later). Relating back to your work in the last section, since d never assigns negative values, this means that
$$AB \prec CD \implies |AB| < |CD|,$$
$$AB \succ CD \implies |AB| > |CD|.$$

It is up to us to build a distance function that meets all three of these requirements. The rest of this chapter is devoted to doing just that.



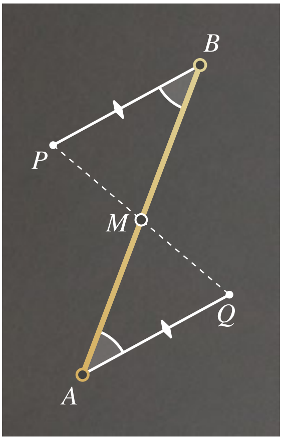
The additivity condition for d.

Divide and combine: the dyadic points

With those conditions in mind, let's start building the distance function d. The picture that I like to keep in my mind as I'm doing this is that simple distance measuring device: the good old-fashioned ruler. Not a metric ruler mind you, but an English ruler with inches on it. Here is one way that you can classify the markings on the ruler. You have the $1''$ mark. That distance is halved, and halved, and halved again to get the $1/2''$, $1/4''$, and $1/8''$ marks. Depending upon the precision of the ruler, there may be $1/16''$ or $1/32''$ markings as well. All the other marks on the ruler are multiples of these. Well, that ruler is the blueprint for how we are going to build the skeleton of d. First of all, because of condition (1), $d(O,O) = 0$. Now take a step along r to another point. Any point is fine– like the inch mark on the ruler, it sets the unit of measurement. Call this point P_0 and define $d(O,P_0) = 1$. Now, as with the ruler, we want to repeatedly halve OP_0. That requires a little theory.

DEF: MIDPOINT
A point M on a segment AB is the *midpoint* of AB if $AM \simeq MB$.

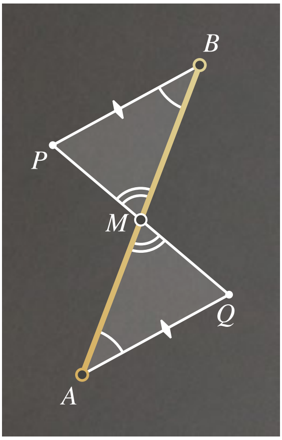
THM: EXISTENCE, UNIQUENESS OF MIDPOINTS
Every segment has a unique midpoint.



Proof. *Existence*. Given the segment AB, choose a point P which is *not* on $\leftarrow AB \rightarrow$. According to the Angle and Segment Construction Axioms, there is a point Q on the opposite side of $\leftarrow AB \rightarrow$ from P so that $\angle ABP \simeq \angle BAQ$ (that's the angle construction part) and so that $BP \simeq AQ$ (that's the segment construction part). Since P and Q are on opposite sides of $\leftarrow AB \rightarrow$, the segment PQ intersects it. Call that point of intersection M. I claim that M is the midpoint of AB. Why? Well, compare $\triangle MBP$ and $\triangle MAQ$.

In those triangles

$$\begin{array}{ll} \angle AMQ \simeq \angle BMP & \text{(vertical angles)} \\ \angle MAQ \simeq \angle MBP & \text{(by construction)} \\ BP \simeq AQ & \text{(by construction)} \end{array}$$



so, by A·A·S, they must be congruent triangles. That means that $AM \simeq MB$. It is worth noting that the midpoint of AB has to be between A and B. If it weren't, one of two things would have to happen:

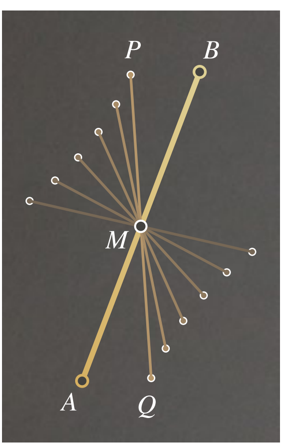
$$\begin{array}{l} M * A * B \implies MA \prec MB, \text{ or} \\ A * B * M \implies MA \succ MB, \end{array}$$

and either way, the segments MA and MB couldn't be congruent.

Uniqueness. Suppose that a segment AB actually had two midpoints. Let's call them M_1 and M_2, and just for the sake of convenience, let's say that they are labeled so that they are ordered as

$$A * M_1 * M_2 * B.$$

There are many choices for P, but they each lead to the same midpoint because a segment can have only one midpoint.



Since $A * M_1 * M_2$, $AM_1 \prec AM_2$. Since $M_1 * M_2 * B$, $BM_2 \prec BM_1$. But now M_2 is a midpoint, so $AM_2 \simeq BM_2$. Let's put that together

$$AM_1 \prec AM_2 \simeq BM_2 \prec BM_1.$$

In the last section you proved that $\prec$ is transitive. This would imply that $AM_1 \prec BM_1$ which contradicts the fact that M_1 is a midpoint. Hence a segment cannot have two distinct midpoints. $\square$

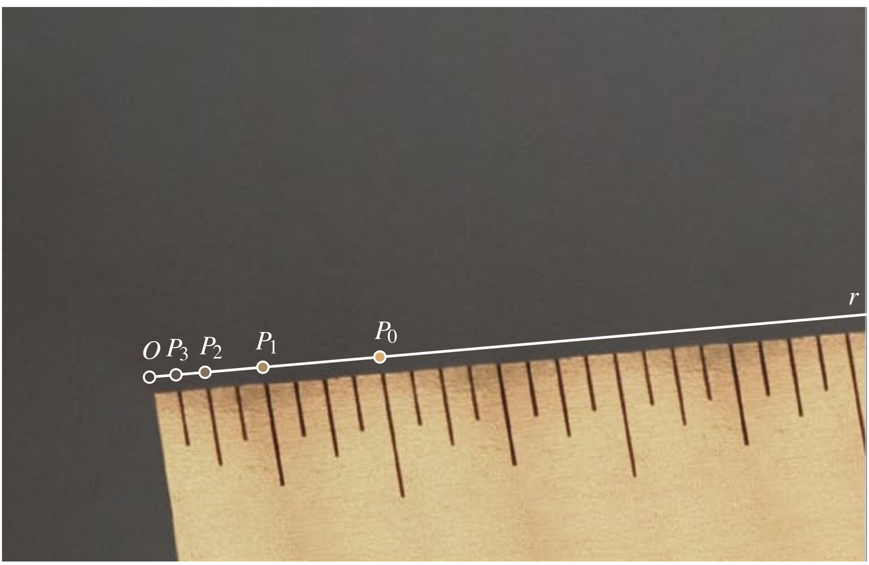
Let's go back to OP_0. We now know that it has a unique midpoint. Let's call that point P_1. In order for the distance function d to satisfy condition (3),

$$|OP_1| + |P_1P_0| = |OP_0|.$$

But OP_1 and P_1P_0 are congruent, so in order for d to satisfy condition (2), they have to be the same length. Therefore $2|OP_1| = 1$ and so $|OP_1| = 1/2$. Repeat. Take OP_1, and find its midpoint. Call it P_2. Then

$$|OP_2| + |P_2P_1| = |OP_1|.$$

Again, OP_2 and P_2P_1 are congruent, so the must be the same length. Therefore $2|OP_2| = 1/2$, and so $|OP_2| = 1/4$. By repeating this process over and over, you can identify the points P_n which are distances of $1/2^n$ from O.



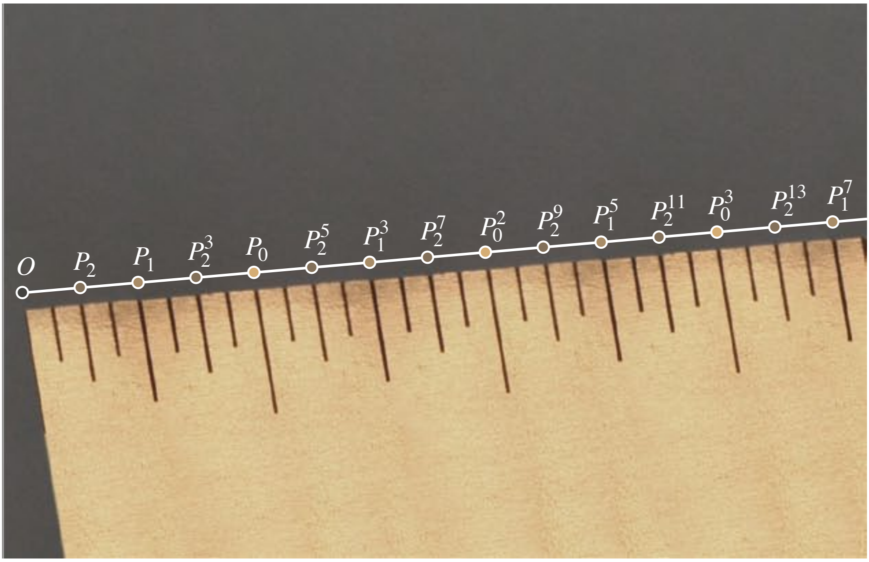
With the points P_n as building blocks, we can start combining segments of lengths $1/2^n$ to get to other points. In fact, we can find a point whose distance from O is $m/2^n$ for any positive integers m and n. It is just a matter of chaining together enough congruent copies of OP_n as follows. Begin with the point P_n. By the first axiom of congruence, there is a point P_n^2 on the opposite side of P_n from O so that $P_nP_n^2 \simeq OP_n$. And there is a point P_n^3 on the opposite side of P_n^2 from P_n so that $P_n^2P_n^3 \simeq OP_n$. And a point P_n^4 on the opposite side of P_n^3 from P_n^2 so that $P_n^3P_n^4 \simeq OP_n$. And so on. This can be continued until m segments are chained together stretching from O to a point which we will label P_n^m. In order for the distance function to satisfy the additivity condition (3),

$$|OP_n^m| = |OP_n| + |P_nP_n^2| + |P_n^2P_n^3| + \cdots + |P_n^{m-1}P_n^m|.$$

All of these segments are congruent, though, so they have to be the same length (for condition (2)), so

$$|OP_n^m| = m \cdot |OP_n| = m \cdot 1/2^n = m/2^n.$$

Rational numbers whose denominator can be written as a power of two are called dyadic rationals. In that spirit, I will call these points the dyadic points of r.



Fill the Hole

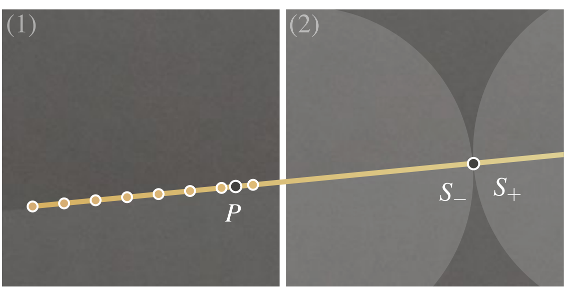
There are plenty of real numbers that aren't dyadic rationals though, and there are plenty of points on r that aren't dyadic points. How can we measure the distance from O to them? For starters, we are not going to be able to do this without the last two axioms of neutral geometry.

These last two axioms, the axioms of continuity, are a little more technical than any of the previous ones. The first says that you can get to any point on a line if you take enough steps. The second, which is inspired by Dedekind's construction of the real numbers, says that there are no gaps in a line.

THE AXIOMS OF CONTINUITY

Ct1 *Archimedes' Axiom* If AB and CD are any two segments, there is some positive integer n such that n congruent copies of CD constructed end-to-end from A along the ray $AB\rightarrow$ will pass beyond B.

Ct2 *Dedekind's Axiom* Let $\mathbb{S}_<$ and $\mathbb{S}_\geq$ be two nonempty subsets of a line ℓ satisfying: (i) $\mathbb{S}_< \cup \mathbb{S}_\geq = \ell$; (ii) no point of $\mathbb{S}_<$ is between two points of $\mathbb{S}_\geq$; and (iii) no point of $\mathbb{S}_\geq$ is between two points of $\mathbb{S}_<$. Then there is a unique point O on ℓ such that for any two other points P_1 and P_2 with $P_1 \in \mathbb{S}_<$ and $P_2 \in \mathbb{S}_\geq$ then $P_1 * O * P_2$



(1) *Archimedes: Given enough steps, P will be passed.*

(2) *Dedekind: There is a point between any two separated partitions of a line.*

It is time to get back to the issue of distance on the ray r. So let P be a point on r. Even if P is not itself a dyadic point, it is surrounded by dyadic points. In fact, there are so many dyadic points crowding P, that the distance from O to P can be estimated to any level of precision using nearby dyadic points. For instance, suppose we consider just the dyadic points whose denominator can be written as 2^0:

$$S_0 = \{O, P_0^1, P_0^2, P_0^3, \ldots\}.$$

By the Archimedean Axiom, eventually these points will lie beyond P. If we focus our attention on the one right before P, say $P_0^{m_0}$, and the one right after, $P_0^{m_0+1}$, then

$$O * P_0^{m_0} * P * P_0^{m_0+1}.$$

We can compare the relative sizes of the segments

$$OP_0^{m_0} \prec OP \prec OP_0^{m_0+1}$$

and so, if our distance is going to satisfy condition (3),

$$\begin{aligned} |OP_0^{m_0}| &< |OP| < |OP_0^{m_0+1}| \\ m_0 &< |OP| < m_0 + 1 \end{aligned}$$

Not precise enough for you? Replace S_0, with S_1, the set of dyadic points whose denominator can be written as 2^1:

$$S_1 = \{O, P_1, P_1^2 = P_0, P_1^3, P_1^4 = P_0^2, \ldots\}.$$

Again, the Archimedean Axiom guarantees that eventually the points in S_1 will pass beyond P. Let $P_1^{m_1}$ be the last one before that happens. Then

$$O * P_1^{m_1} * P * P_1^{m_1+1}$$

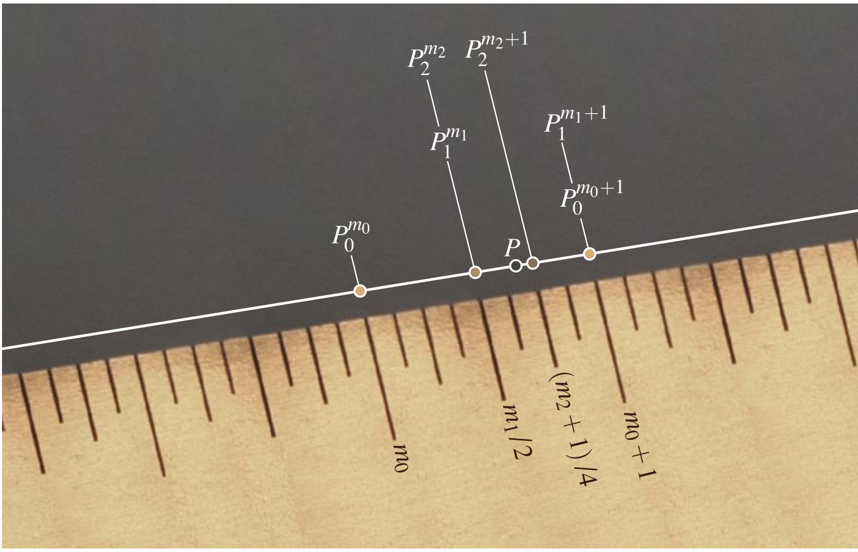
so

$$\begin{aligned} |OP_1^{m_1}| &< |OP| < |OP_1^{m_1+1}| \\ m_1/2 &< |OP| < (m_1 + 1)/2 \end{aligned}$$

and this gives $|OP|$ to within an accuracy of 1/2.

Continuing along in this way, you can use S_2, dyadics whose denominator can be written as 2^2, to approximate $|OP|$ to within $1/4$, and you can use S_3, dyadics whose denominator can be written as 2^3, to approximate $|OP|$ to within $1/8$. Generally speaking, the dyadic rationals in S_n provide an upper and lower bound for $|OP|$ which differ by $1/2^n$. As n goes to infinity, $1/2^n$ goes to zero, forcing the upper and lower bounds to come together at a single number. This number is going to have to be $|OP|$. Now you don't really need both the increasing and decreasing sequences of approximations to define $|OP|$. After all, they both end up at the same number. Here is the description of $|OP|$ using just the increasing sequence: for each positive integer n, let $P_n^{m_n}$ be the last point in the list S_n which is between O and P. In order for the distance function to satisfy condition (3), we must set

$$|OP| = \lim_{n\to\infty} |OP_n^{m_n}| = \lim_{n\to\infty} m_n/2^n.$$



Capturing a non-dyadic point between two sequences of dyadic points.

Now do it in reverse

Every point of r now has a distance associated with it, but is there a point at every possible distance? Do we know, for instance, that there is a point at exactly a distance of 1/3 from O? The answer is yes– it is just a matter of reversing the distance calculation process we just described and using the Dedekind Axiom. Let's take as our prospective distance some positive real number x. For each integer $n \geq 0$, let $m_n/2^n$ be the largest dyadic rational less than x whose denominator can be written as 2^n and let $P_n^{m_n}$ be the corresponding dyadic point on r. Now we are going to define two sets of points:

$\mathbb{S}_<$: all the points of r that lie between O and any of the $P_n^{m_n}$, together with all the points of r^{op}.

$\mathbb{S}_\geq$: all of the remaining points of r.

So $\mathbb{S}_<$ contains a sequence of dyadic rationals increasing to x

$$\{P_0^{m_0}, P_1^{m_1}, P_2^{m_2}, P_3^{m_3}, \ldots\},$$

and $\mathbb{S}_{\geq}$ contains a sequence of dyadic rationals decreasing to x

$$\{P_0^{m_0+1}, P_1^{m_1+1}, P_2^{m_2+1}, P_3^{m_3+1}, \ldots\}.$$

Together $\mathbb{S}_{<}$ and $\mathbb{S}_{\geq}$ contain all the points of the line through r, but they do not intermingle: no point of $\mathbb{S}_{<}$ is between two of $\mathbb{S}_{\geq}$ and no point of $\mathbb{S}_{\geq}$ is between two of $\mathbb{S}_{<}$. According to the Dedekind Axiom, then, there is a unique point P between $\mathbb{S}_{<}$ and $\mathbb{S}_{\geq}$. Now let's take a look at how far P is from O. For all n,

$$\begin{aligned} OP_n^{m_n} &\prec OP \prec OP_n^{m_n+1} \\ |OP_n^{m_n}| &< |OP| < |OP_n^{m_n+1}| \\ m_n/2^n &< |OP| < (m_n+1)/2^n \end{aligned}$$

As n goes to infinity, the interval between these two consecutive dyadics shrinks – ultimately, the only point left is x. So $|OP| = x$.

Example: dyadics approaching 1/3

Finding a dyadic sequence approaching a particular number can be tricky business. Finding such a sequence approaching 1/3 is easy, though, as long as you remember the geometric series formula

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{if } |x| < 1.$$

With a little trial and error, I found that by plugging in $x = 1/4$,

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{4}{3}.$$

Subtracting one from both sides gives an infinite sum of dyadics to 1/3, and we can extract the sequence from that

$$\begin{aligned} &\frac{1}{4} = 0.25 \\ &\frac{1}{4} + \frac{1}{16} = \frac{5}{16} = 0.3125 \\ &\frac{1}{4} + \frac{1}{16} + \frac{1}{64} = \frac{21}{64} = 0.32825 \end{aligned}$$

Segment addition, redux

For any two points P and Q, there is a unique segment OR on the ray r which is congruent to PQ. Define $d(P,Q) = |OR|$. With this setup, our distance function will satisfy conditions (1) and (2). That leaves condition (3)– a lot of effort went into trying to build d so that condition would be satisfied, but it is probably a good idea to make sure that it actually worked. Let's close out this lesson with two theorems that do that.

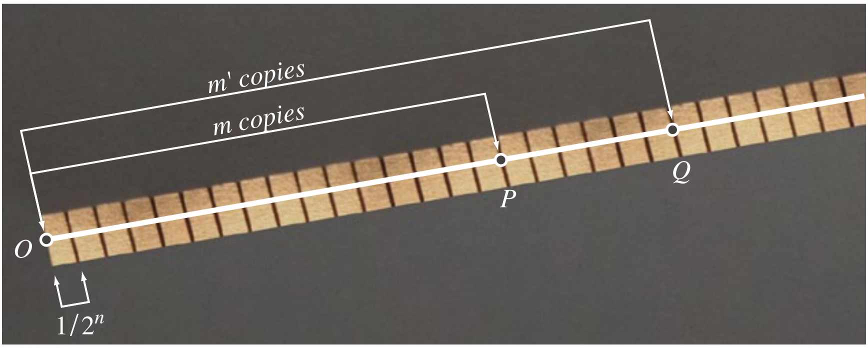
THM: A FORMULA FOR DISTANCE ALONG A RAY
If P and Q are points on r, with $|OP| = x$ and $|OQ| = y$, and if P is between O and Q, then $|PQ| = y - x$.

Proof. If both P and Q are dyadic points, then this is fairly easy. First you are going to express their dyadic distances with a common denominator:

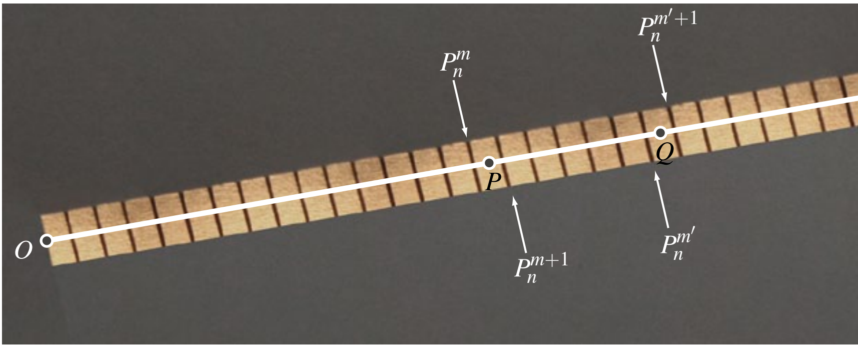
$$|OP| = m/2^n \quad |OQ| = m'/2^n.$$

Then OP is built from m segments of length $1/2^n$ and OQ is built from m' segments of length $1/2^n$. To get $|PQ|$, you simply have to take the m segments from the m' segments– so $|PQ|$ is made up of $m' - m$ segments of length $1/2^n$. That is

$$|PQ| = (m' - m) \cdot \frac{1}{2^n} = y - x.$$



Measuring the distance between two dyadic points.



Measuring the distance between two non-dyadic points.

If one or both of P and Q are not dyadic, then there is a bit more work to do. In this case, P and Q are approximated by a sequence of dyadics $P_n^{m_n}$ and $P_n^{m'_n}$ where

$$\lim_{n\to\infty} \frac{m_n}{2^n} = x \quad \& \quad \lim_{n\to\infty} \frac{m'_n}{2^n} = y.$$

Now we can trap $|PQ|$ between dyadic lengths:

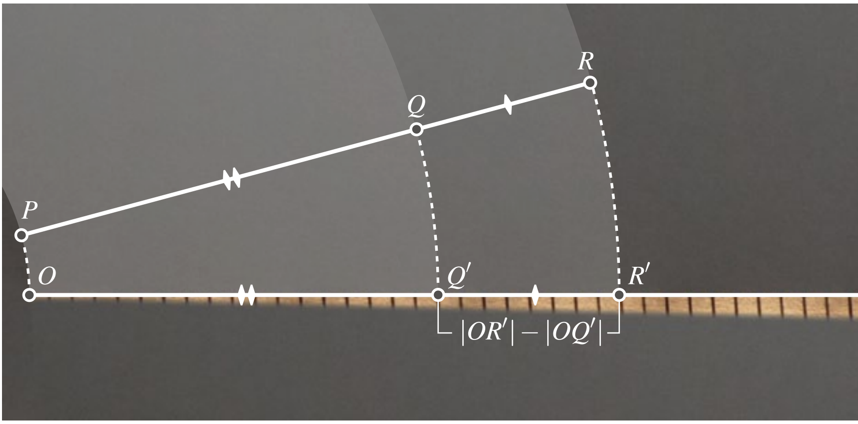
$$P_n^{m_n+1} P_n^{m'_n} \prec PQ \prec P_n^{m_n} P_n^{m'_n+1}$$
$$|P_n^{m_n+1} P_n^{m'_n}| < |PQ| < |P_n^{m_n} P_n^{m'_n+1}|$$
$$\frac{m'_n - m_n - 1}{2^n} < |PQ| < \frac{m'_n + 1 - m_n}{2^n}$$

As n approaches infinity, $|PQ|$ is stuck between two values both of which are approaching $y - x$. □

Now while this result only gives a formula for lengths of segments on the ray r, it is easy to extend it to a formula for lengths of segments on the line containing r. In fact, this is one of the exercises for this lesson. The last result of this lesson is a reinterpretation of the Segment Addition Axiom in terms of distance, and it confirms that the distance we have constructed does satisfy condition (3).

THM: SEGMENT ADDITION, THE MEASURED VERSION
If $P * Q * R$, then $|PQ| + |QR| = |PR|$.

Proof. The first step is to transfer the problem over to r so that we can start measuring stuff. So locate Q' and R' on r so that:



$$O * Q' * R', \quad PQ \simeq OQ', \quad QR \simeq Q'R'.$$

According to the Segment Addition Axiom, this means that $PR \simeq OR'$. Now we can use the last theorem,

$$|QR| = |Q'R'| = |OR'| - |OQ'| = |PR| - |PQ|.$$

Just solve that for $|PR|$ and you've got it. □

Exercises

1. Our method of measuring distance along a ray r can be extended to the rest of the line. In our construction each point on r corresponds to a positive real number (the distance from O to that point). Suppose that P is a point on r^{op}. There is a point Q on r so that $OP \simeq OQ$. If x is the positive real number associated with Q, then we want to assign the negative number $-x$ to P. Now suppose that P_1 and P_2 are any two points on the line and x and y are the associated real numbers. Show that
$$d(P_1, P_2) = |x - y|.$$

2. Write $1/7$, $1/6$, and $1/5$ as an infinite sum of dyadic rationals.

3. Since writing this, it has come to my attention (via Greenberg's book [1]) that Archimedes' Axiom is actually a consequece of Dedekind's Axiom. You can prove this yourself as follows. If Archimedes were not true, then there would be some point on a ray that could not be reached by via end-to-end copies of a segment. In that case, the ray can be divided into two sets: one consisting of the points that can be reached, the other of the points that cannot. By including the opposite ray in with the set of points that can be reached, you get a partition of a line into two sets. Prove that these sets form a Dedekind cut of the line. Then by Dedekind's Axiom there is a point between them. Now consider what would happen if you took one step forward or backward from this point.

References

[1] Marvin J. Greenberg. *Euclidean and Non-Euclidean Geometries: Development and History*. W.H. Freeman and Company, New York, 4th edition, 2008.

8. NARROWER AND WIDER
ANGLE COMPARISON

These next two chapters are devoted to developing a measurement system for angles. It's really not that different from what we did in the last two chapters and again I would like to divide up the work so I don't feel like I am doing everything by myself. This time I will prove the results about the synthetic comparison of angles and I will let you prove the results which ultimately lead to the degree system of angle measurement.

Synthetic angle comparison

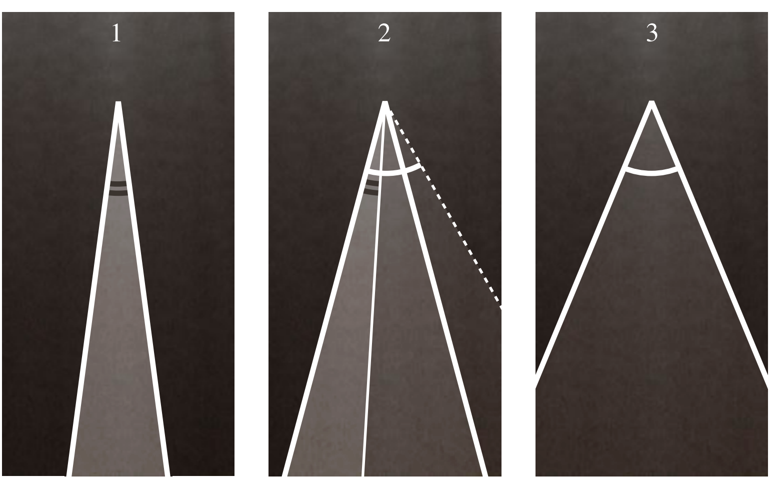
The first step is to develop a way to compare angles so that you can look at two angles and say that one is smaller or larger than the other. I gave these definitions back in lesson 4, but in the interest of keeping everything together, and to introduce some notation, here they are again.

DEF: SMALLER AND LARGER ANGLES

Given angles $\angle ABC$ and $\angle A'B'C'$, label $C^\star$ on the same side of AB as C so that $\angle ABC^\star \simeq \angle A'B'C'$.

$\prec$ If $C^\star$ is in the interior of $\angle ABC$, then $\angle A'B'C'$ is *smaller than* $\angle ABC$, written $\angle A'B'C' \prec \angle ABC$.

$\succ$ If $C^\star$ is in the exterior of $\angle ABC$, then $\angle A'B'C'$ is *larger than* $\angle ABC$, written $\angle A'B'C' \succ \angle ABC$.



$\angle 1 \prec \angle 2$ $\angle 3 \succ \angle 2$

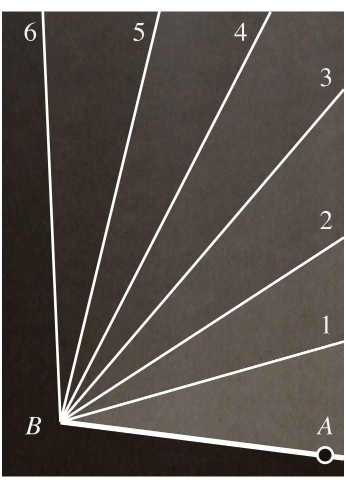
In addition, the results of this section depend upon two results we proved a while ago.

THM: ORDERING RAYS

Given $n \geq 2$ rays with a common basepoint B which are all on the same side of the line $\overleftrightarrow{AB}$ through B, there is an ordering of them:

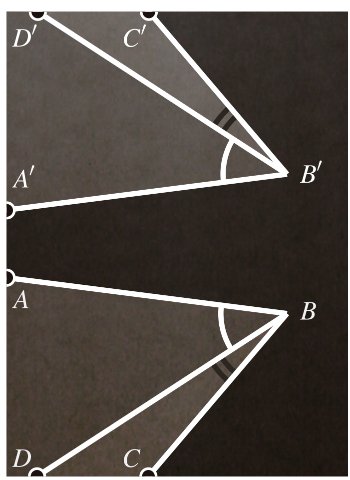
$$r_1,\ r_2,\ \ldots, r_n$$

so that if $i < j$ then r_i is in the interior of the angle formed by $\overrightarrow{BA}$ and r_j.



THM: CONGRUENCE AND ANGLE INTERIORS

Given $\angle ABC \simeq \angle A'B'C'$ and that the point D is in the interior of $\angle ABC$. Suppose that D' is located on the same side of $\overleftrightarrow{AB}$ as C so that $\angle ABD \simeq \angle A'B'D'$. Then D' is in the interior of $\angle A'B'C'$.



As with the segment comparison definitions, there is a potential issue with the definitions of $\prec$ and $\succ$. What if we decided to construct $C^\star$ off of $BC\rightarrow$ instead of $BA\rightarrow$? Since $\angle ABC = \angle CBA$, and since we are interested in comparing the angles themselves, this notion of larger or smaller should not depend upon which ray we are building from. The next theorem tells us not to worry.

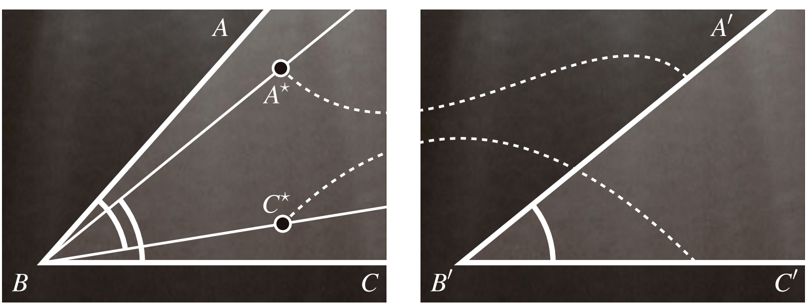
THM: $\prec$ AND $\succ$ ARE WELL DEFINED
Given $\angle ABC$ and $\angle A'B'C'$, label:

$C^\star$– a point on the same side of AB as C for which $\angle ABC^\star \simeq \angle A'B'C'$

$A^\star$– a point on the same side of BC as A for which $\angle CBA^\star \simeq \angle A'B'C'$.

Then $C^\star$ is in the interior of $\angle ABC$ if and only if $A^\star$ is.

Proof. This is really a direct corollary of the "Congruence and Angle Interiors" result from lesson 3. You see, that is exactly what we have here: $\angle ABC \simeq \angle ABC$ and $\angle A^\star BC \simeq \angle ABC^\star$ and $C^\star$ is on the same side of AB as C, so if $A^\star$ is in the interior of $\angle ABC$, then $C^\star$ must be too. Conversely, $A^\star$ is on the same side of BC as A, so if $C^\star$ is in the interior, then $A^\star$ must be too. $\square$



When comparing angles, it doesn't matter which ray is used as the "base".

Now let's take a look at some of the properties of synthetic angle comparison. I am focusing on the $\prec$ version of these properties: the $\succ$ version should be easy enough to figure out from these. There is nothing particularly elegant about these proofs. They mainly rely upon the two theorems listed above.

THM: TRANSITIVITY OF $\prec$

$\prec\prec$ If $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$ and $\angle A_2B_2C_2 \prec \angle A_3B_3C_3$, then $\angle A_1B_1C_1 \prec \angle A_3B_3C_3$.

$\simeq\prec$ If $\angle A_1B_1C_1 \simeq \angle A_2B_2C_2$ and $\angle A_2B_2C_2 \prec \angle A_3B_3C_3$, then $\angle A_1B_1C_1 \prec \angle A_3B_3C_3$.

$\prec\simeq$ If $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$ and $\angle A_2B_2C_2 \simeq \angle A_3B_3C_3$, then $\angle A_1B_1C_1 \prec \angle A_3B_3C_3$.

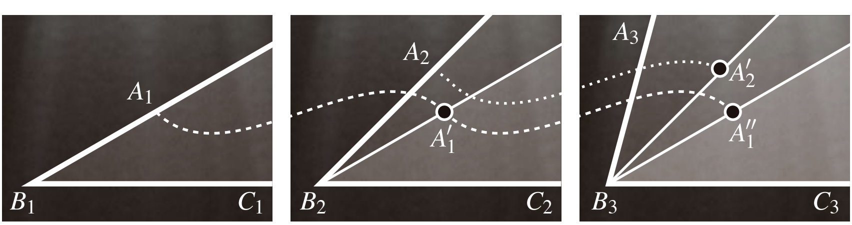
Proof. Let me just take the first of these statements since the other two are easier. Most of the proof is just getting points shifted into a useful position.

1. Copy the first angle into the second: since $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$, there is a point A_1' in the interior of $\angle A_2B_2C_2$ so that $\angle A_1B_1C_1 \simeq \angle A_1'B_2C_2$.

2. Copy the second angle in to the third: since $\angle A_2B_2C_2 \prec \angle A_3B_3C_3$, there is a point A_2' in the interior of $\angle A_3B_3C_3$ so that $\angle A_2B_2C_2 \simeq \angle A_2'B_3C_3$.

3. Copy the first angle to the third (although we don't know quite as much about this one): pick a point A_1'' on the same side of B_3C_3 as A_1 so that $A_1''B_3C_3 \simeq A_1B_1C_1$.

Now we can get down to business. "Congruence and Angle Interiors": since A_1' is in the interior of $\angle A_2B_2C_2$, A_1'' has to be in the interior of $\angle A_2'B_3C_3$. "Ordering rays": since $B_3A_1''\rightarrow$ is in the interior of $\angle A_3B_3A_2'$, and since $B_3A_2'\rightarrow$ is in the interior of $\angle A_3B_3C_3$, this means that $B_3A_1''\rightarrow$ has to be in the interior of $\angle A_3B_3C_3$. Therefore $\angle A_1B_1C_1 \prec \angle A_3B_3C_3$. □

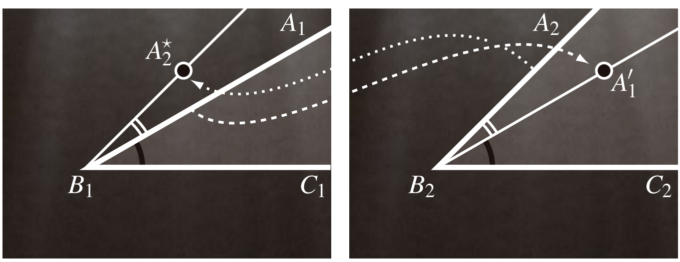


The transitivity of $\prec$.

THM: SYMMETRY BETWEEN $\prec$ AND $\succ$

For any two angles $\angle A_1B_1C_1$ and $\angle A_2B_2C_2$, $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$ if and only if $\angle A_2B_2C_2 \succ \angle A_1B_1C_1$.

Proof. This is a direct consequence of the "Congruence and Angle Interiors" theorem. Suppose that $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$. Then there is a point A_1' in the interior of $\angle A_2B_2C_2$ so that $\angle A_1B_1C_1 \simeq \angle A_1'B_2C_2$. Moving back to the first angle, there is a point $A_2^\star$ on the opposite side of A_1B_1 from C_1 so that $\angle A_1B_1A_2^\star \simeq \angle A_1'B_2A_2$. By angle addition, $\angle A_2^\star B_1C_1 \simeq \angle A_2B_2C_2$, and since $A_2^\star$ is not in the interior of $\angle A_1B_1C_1$, that means $\angle A_2B_2C_2 \succ \angle A_1B_1C_1$. The other direction in this proof works very similarly so I won't go through it. $\square$



Relating $\prec$ and $\succ$

THM: ORDERING FOUR RAYS

If A_2 and C_2 are in the interior of $\angle A_1BC_1$, then $\angle A_2BC_2 \prec \angle A_1BC_1$.

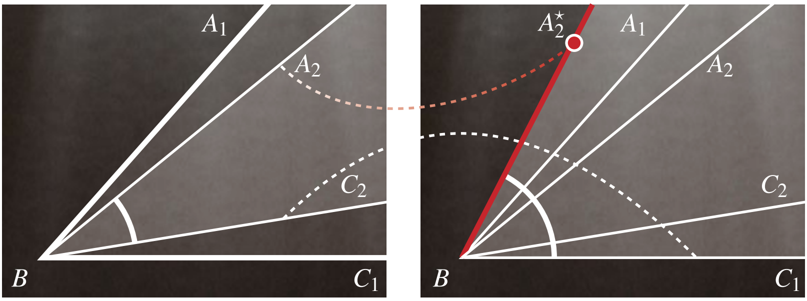
Proof. Locate $A_2^\star$ on the same side of $\overleftrightarrow{BC_1}$ as A_1 so that

$$\angle A_2^\star BC_1 \simeq \angle A_2BC_2.$$

Then the question is– is $A_2^\star$ in the interior of $\angle A_1BC_1$? Well, let's suppose that it isn't. Then

$$\angle A_2BC_2 \prec \angle A_2^\star BC_2 \prec \angle A_2^\star BC_1.$$

Since we have established that $\prec$ is transitive, that means $\angle A_2BC_2 \prec \angle A_2^\star BC_1$. But this cannot be– those two angles are supposed to be congruent. Hence $A_2^\star$ has to be in the interior of $\angle A_1BC_1$, and so $\angle A_2BC_2 \prec \angle A_1BC_1$. $\square$



Proof by contradiction of the "Ordering Four Rays" Theorem.

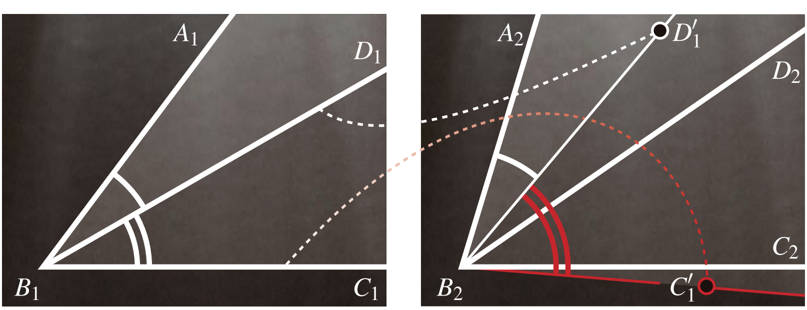
THM: ADDITIVITY OF $\prec$

Suppose that D_1 lies in the interior of $\angle A_1B_1C_1$ and that D_2 lies in the interior of $\angle A_2B_2C_2$. If $\angle A_1B_1D_1 \prec \angle A_2B_2D_2$ and $\angle D_1B_1C_1 \prec \angle D_2B_2C_2$, then $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$.

Proof. Find D_1' in the interior of $\angle A_2B_2D_2$ so that $\angle A_2B_2D_1' \simeq \angle A_1B_1D_1$. Find C_1' on the opposite side of $\leftarrow B_2D_1' \rightarrow$ from A_2 so that $\angle D_1'B_2C_1' \simeq \angle D_1B_1C_1$. By angle addition, $\angle A_2B_2C_1' \simeq \angle A_1B_1C_1$, so the question is whether or not C_1' is in the interior of $\angle A_2B_2C_2$. Well, if it was not, then by the previous theorem

$$\angle D_2B_2C_2 \prec \angle D_1'B_2C_1' \quad \Longrightarrow \quad \angle D_2B_2C_2 \prec \angle D_1B_1C_1.$$

That is a contradiction (the angles were constructed to be congruent), so C_1' will have to lie in the interior of $\angle A_2B_2C_2$, and so $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$. □

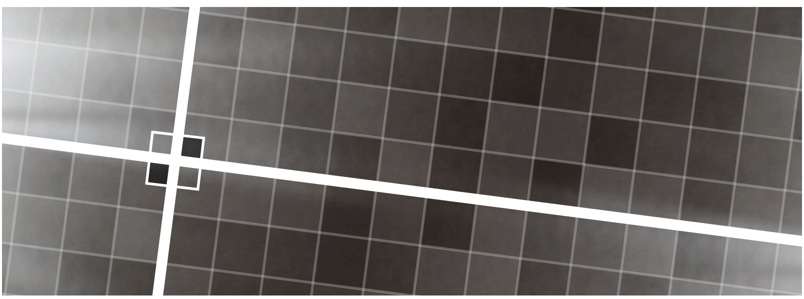


The proof by contradiction of the additivity of $\prec$

Right angles

Distance and segment length is based upon a completely arbitrary segment to determine unit length. Angle measure is handled differently– a specific angle is used as the baseline from which the rest is developed (although, at least in the degree measurement system, that angle is then assigned a pretty random measure). That angle is the right angle.

DEF: RIGHT ANGLE
A right angle is an angle which is congruent to its own supplement.

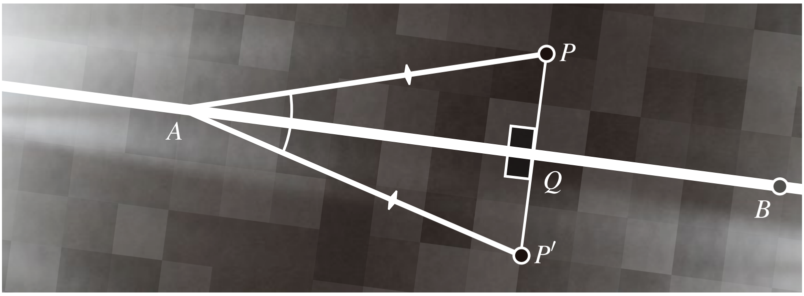


Right angles. In diagrams, squares angle markers are often used to indicate that an angle is right.

Now I didn't mention it at the time, but we have already stumbled across right angles once, in the proof of the S·S·S theorem. But it ought to be stated again, that

THM: RIGHT ANGLES, EXISTENCE
Right angles do exist.

Proof. We will prove that right angles exist by constructing one. Start with a segment AB. Now choose a point P which is not on the line $\leftarrow AB \rightarrow$. If $\angle PAB$ is congruent to its supplement, then it is a right angle, and that's it. If $\angle PAB$ is not congruent to its supplement (which is really a lot more likely), then there is a little more work to do. Thanks to the Segment and Angle Construction Axioms, there is a point P' on the opposite side of $\leftarrow AB \rightarrow$ from P so that $\angle P'AB \simeq \angle PAB$ (angle construction) and $AP' \simeq AP$



Proof of existence by construction.

(segment construction). Since P and P' are on opposite sides of $\leftarrow AB \rightarrow$, the segment PP' has to intersect the line $\leftarrow AB \rightarrow$. Call that point of intersection Q. With that construction,

$$PA \simeq P'A \quad \angle PAQ \simeq \angle P'AQ \quad AQ \simeq AQ$$

so by S·A·S triangle congruence theorem, $\triangle PAQ \simeq \triangle P'AQ$. Out of those two triangles, the relevant congruence is between the two angles that share the vertex Q: $\angle AQP \simeq \angle AQP'$. These angles are supplements. They are congruent. By definition, they are right angles. $\square$

Okay, so they are out there. But how many are there? The next result is something like a uniqueness statement– that there is really only one right angle “modulo congruence”.

THM: RIGHT ANGLES AND CONGRUENCE

Suppose that $\angle ABC$ is a right angle. Then $\angle A'B'C'$ is a right angle if and only if it is congruent to $\angle ABC$.

Proof. This is an “if and only if” statement, and that means that there are two directions to prove.

$\Longrightarrow$ If $\angle A'B'C'$ is a right angle, then $\angle A'B'C' \simeq \angle ABC$.

$\Longleftarrow$ If $\angle A'B'C' \simeq \angle ABC$, then $\angle A'B'C'$ is a right angle.

$\Longrightarrow$ To start, let's go ahead and mark a few more points so that we can refer to the supplements of these angles. Mark the points

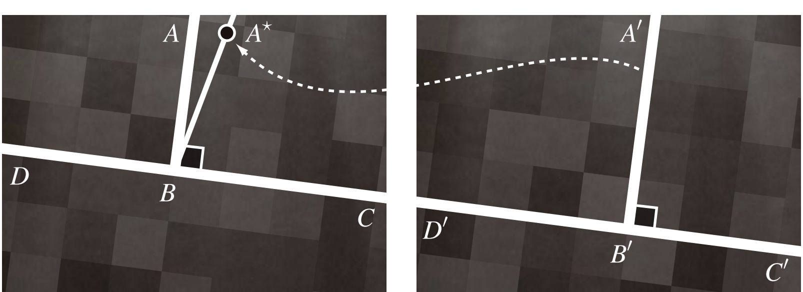
D on $\leftarrow BC\rightarrow$ so that $D*B*C$ and

D' on $\leftarrow B'C'\rightarrow$ so that $D'*B'*C'$.

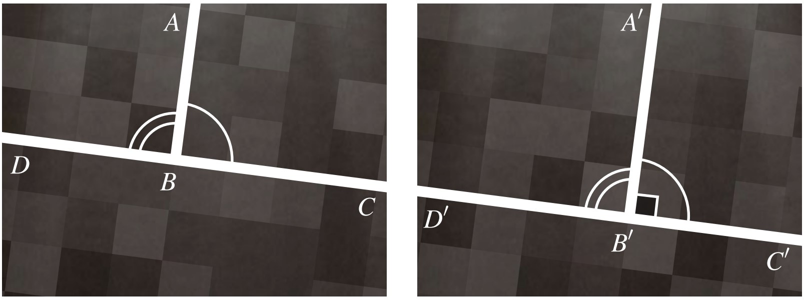
Therefore $\angle ABC$ and $\angle ABD$ are a supplementary pair, as are $\angle A'B'C'$ and $\angle A'B'D'$. Now suppose that both $\angle ABC$ and $\angle A'B'C'$ are right angles. Thanks to the Angle Construction Axiom, it is possible to build a congruent copy of $\angle A'B'C'$ on top of $\angle ABC$: there is a ray $BA^\star\rightarrow$ on the same side of BC as A so that $\angle A^\star BC \simeq \angle A'B'C'$. Earlier we proved that the supplements of congruent angles are congruent, so that means $\angle A^\star BD \simeq \angle A'B'D'$. How, though, does $\angle A^\star BC$ compare to $\angle ABC$? If $BA^\star\rightarrow$ and $BA\rightarrow$ are the same ray, then the angles are equal, meaning that $\angle ABC$ and $\angle A'B'C'$ are congruent– which is what we want. But what happens if the two rays are not equal? In that case one of two things can happen: either $BA^\star\rightarrow$ is in the interior of $\angle ABC$, or it is in the interior of $\angle ABD$. Both of these cases are going to leads to essentially the same problem, so let me just focus on the first one. In that case, $A^\star$ is in the interior of $\angle ABC$, so $\angle A^\star BC \prec \angle ABC$, but $A^\star$ is in the exterior of $\angle ABD$, so $\angle A^\star BD \succ \angle ABD$. That leads to a string of congruences and inequalities:

$$\angle A'B'C' \simeq \angle A^\star BC \prec \angle ABC \simeq \angle ABD \prec \angle A^\star BD \simeq \angle A'B'D'.$$

Because of the transitivity of $\prec$ then, $\angle A'B'C' \prec \angle A'B'D'$. This can't be– those two supplements are supposed to be congruent. The second scenario plays out in the same way, with $\succ$ in place of $\prec$. Therefore $BA^\star\rightarrow$ and $BA\rightarrow$ have to be the same ray, and so $\angle A'B'C \simeq \angle ABC$.



Any two right angles are congruent: if one right angle were larger or smaller than another, it could not be congruent to its complement.



If an angle is congruent to a right angle, it is a right angle too.

$\Longleftarrow$ The other direction is easier. Suppose that $\angle A'B'C' \simeq \angle ABC$ and that $\angle ABC$ is a right angle. Let's recycle the points D and D' from the first part of the proof. The angles $\angle A'B'D'$ and $\angle ABD$ are supplementary to congruent angles, so they too must be congruent. Therefore

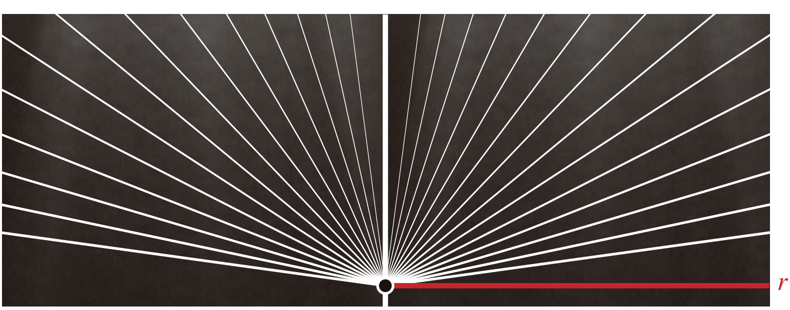
$$\angle A'B'C' \simeq \angle ABC \simeq \angle ABD \simeq \angle A'B'D'.$$

and so we can see that $\angle A'B'C'$ is congruent to its supplement– it must be a right angle. $\square$

With $\prec$ and $\succ$ and with right angles as a point of comparison, we have a way to classify non-right angles.

DEF: ACUTE AND OBTUSE

An angle is *acute* if it is smaller than a right angle. An angle is *obtuse* if it is larger than a right angle.



Rays that form an obtuse angle with r. *Rays that form an acute angle with r.*

Exercises

1. Verify that the supplement of an acute angle is an obtuse angle and that the supplement of an obtuse angle is an acute angle.

2. Prove that an acute angle cannot be congruent to an obtuse angle (and vice versa).

3. Two intersecting lines are *perpendicular* if the angles formed at their intersection are right angles. For any line ℓ and point P, prove that there is a unique line through P which is perpendicular to ℓ. Note that there are two scenarios: P may or may not be on ℓ.

4. Consider two isosceles triangles with a common side: $\triangle ABC$ and $\triangle A'BC$ with $AB \simeq AC$ and $A'B \simeq A'C$. Prove that $\leftarrow AA' \rightarrow$ is perpendicular to $\leftarrow BC \rightarrow$.

5. Two angles are *complementary* if together they form a right angle. That is, if D is in the interior of a right angle $\angle ABC$, then $\angle ABD$ and $\angle DBC$ are complementary angles. Prove that every acute angle has a complement. Prove that if $\angle ABC$ and $\angle A'B'C'$ are congruent acute angles, then their complements are also congruent.

6. Verify that if ℓ_1 is perpendicular to ℓ_2 and ℓ_2 is perpendicular to ℓ_3, then either $\ell_1 = \ell_3$, or ℓ_1 and ℓ_3 are parallel.

9. READER'S SOLO II
ANGLE MEASURE

In this lesson I am going to outline what you need to do to construct the degree measurement system for angles. First, let's talk notation. I think the most common way to indicate the measure of an angle $\angle ABC$ is to write $m(\angle ABC)$. The advantage of that notation is that it draws a clear distinction between an angle and its measure. Of course, the disadvantage is that it is cumbersome, and that any equation with lots of angles measures in it will be cluttered up with m's. At the other extreme, I have noticed that students tend to just write the angle $\angle ABC$ to indicate its measure. Sure, it is just laziness, but I suppose you could pass it off as notational efficiency as well. The obvious disadvantage of this approach is that it completely blurs the distinction between an angle and its measure. I have tried to find the middle ground between these two approaches and I write $(\angle ABC)$ to denote the measure of $\angle ABC$. This notation is not perfect either. I think the biggest problem is that it puts even more pressure on two of the most overused symbols in mathematics, the parentheses.

Now lets talk about what you are going to want in a system of angle measurement. Of course these expectations are going to closely mirror expectations for measures of distance. They are

(1) The measure of an angle should be a positive real number.

(2) Congruent angles should have the same measure. That allows us to focus our investigation on just the angles which are built off of one fixed ray.

(3) If D is in the interior of $\angle ABC$, then

$$(\angle ABC) = (\angle ABD) + (\angle DBC).$$

Therefore, since the measure of an angle has to be positive,

$$\begin{aligned} \angle ABC \prec \angle A'B'C' &\implies (\angle ABC) < (\angle A'B'C') \\ \angle ABC \succ \angle A'B'C' &\implies (\angle ABC) > (\angle A'B'C'). \end{aligned}$$

It is your turn to develop a system of angle measure that will meet those requirements. The first step is to establish the measurement of dyadic angles. To do that, you will have to prove that it is possible to divide an angle in half.

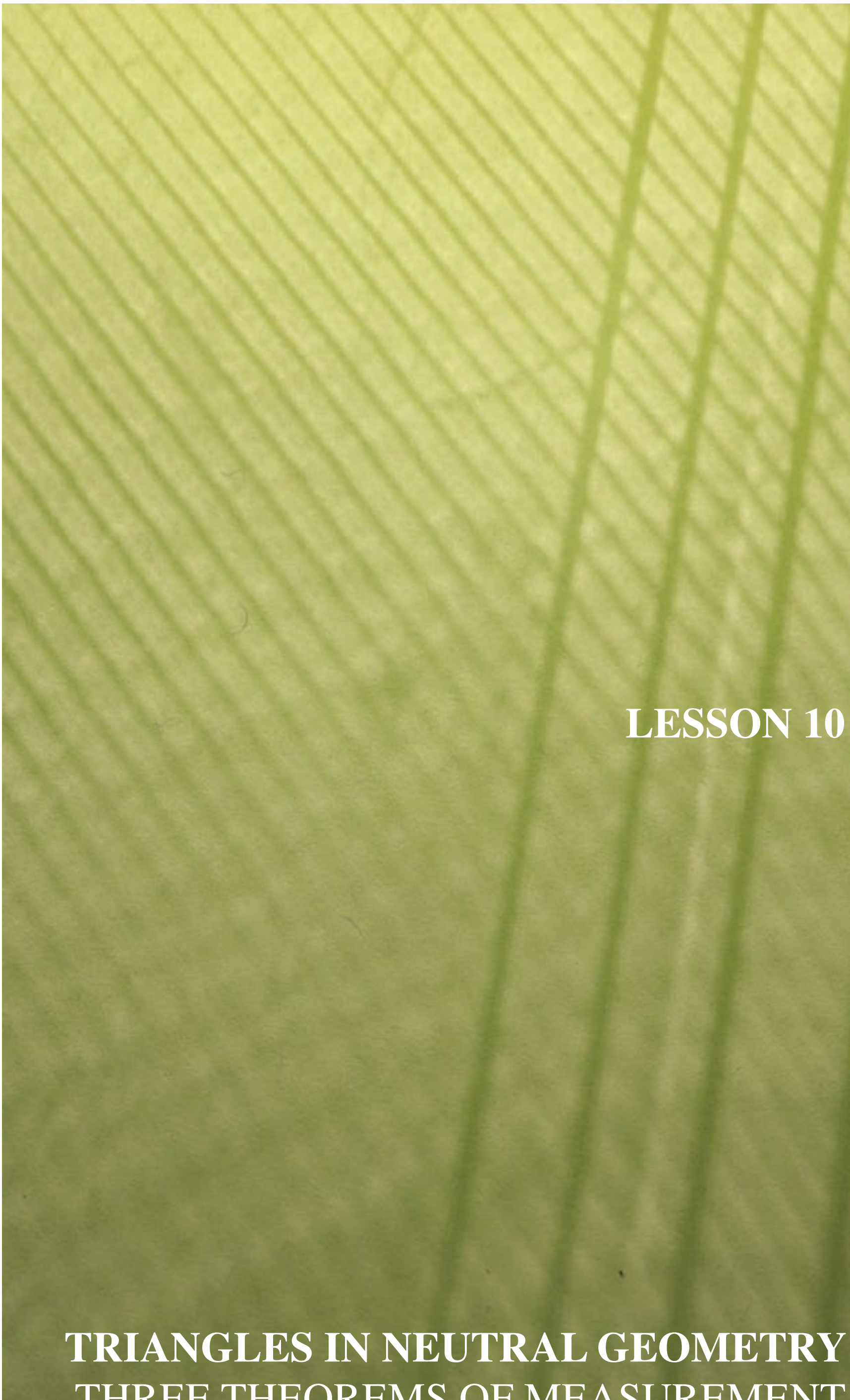
DEF: ANGLE BISECTOR
For any angle $\angle ABC$, there is a unique ray $BD \rightarrow$ in the interior of $\angle ABC$ so that $\angle ABD \simeq \angle DBC$. This ray is called the *angle bisector* of $\angle ABC$.

With segment length, everything begins with an arbitrary segment which is assigned a length of one. With angle measure, everything begins with a right angle which, in the degree measurement system, is assigned a measure of $90°$. From that, your next step is to describe the process of constructing angles with measures $90° \cdot m/2^n$. Here you are going to run into one fundamental difference between angles and segments– segments can be extended arbitrarily, but angles cannot be put together to exceed a straight angle. Therefore segments can be arbitrarily long, but all angles must measure less than $180°$ (since a straight angle is made up of two right angles). It is true that the unit circle in trigonometry shows how you can loop back around to define angles with any real measure, positive or negative, and that is a useful extension in some contexts, but it also creates some problems (the measure of an angle is not uniquely defined, for instance).

Once you have figured out the dyadic angles, you need to fill in the rest. You will want to use a limiting process just like I did in the segment length chapter: this time the key word "interior" will replace the key word "between." Then you will want to turn the question around: for any real number in the interval $(0°, 180°)$ is there an angle with that as its measure? This is where I used the Dedekind Axiom before, by taking a limit of approximating dyadics, and then using the axiom to say that there is a point at that limit. The problem for you is that the Dedekind Axiom applies only to points on a line– it is not about angles (or at least not directly). Nevertheless, you need to find a way to set up approximating dyadic angles, and then you need to find some way to make Dedekind's Axiom apply in this situation.

Finally, with angles measured in this way, you will need to verify the additivity of angle measure:

THM: ANGLE ADDITION, THE MEASURED VERSION
If D is in the interior of $\angle ABC$, then $(\angle ABC) = (\angle ABD) + (\angle DBC)$.



LESSON 10

TRIANGLES IN NEUTRAL GEOMETRY
THREE THEOREMS OF MEASUREMENT

In this lesson we are going to take our newly created measurement systems, our rulers and our protractors, and see what we can tell us about triangles. We will derive three of the most fundamental results of neutral geometry: the Saccheri-Legendre Theorem, the Scalene Triangle Theorem, and the Triangle Inequality.

The Saccheri-Legendre Theorem

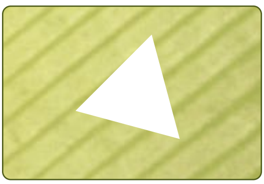
The Saccheri-Legendre Theorem is a theorem about the measures of the interior angles of a triangle. For the duration of this lesson, if $\triangle ABC$ is any triangle, I will call

$$s(\triangle ABC) = (\angle A) + (\angle B) + (\angle C)$$

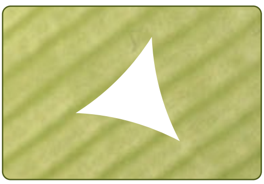
the *angle sum* of the triangle. As you probably know, in Euclidean geometry the angle sum of any triangle is 180°. That is not necessarily the case in neutral geometry, though, so we will have to be content with a less restrictive (and less useful) condition.

THE SACCHERI LEGENDRE THEOREM
For any triangle $\triangle ABC$, $s(\triangle ABC) \leq 180^\circ$.

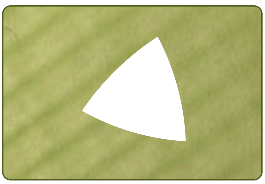
I will prove this result in three parts– two preparatory lemmas followed by the proof of the main theorem.



Euclidean triangle



non-Euclidean triangle



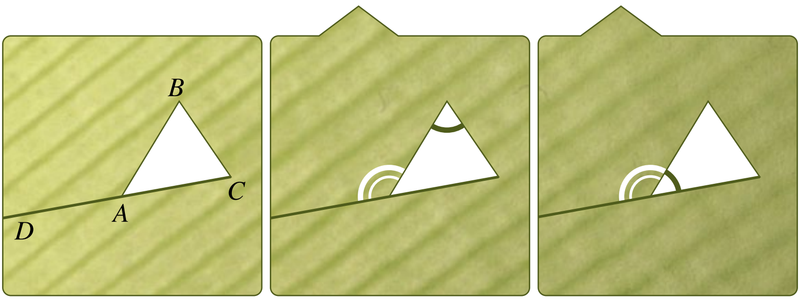
not a neutral triangle

LEMMA ONE

The sum of the measures of any two angles in a triangle is less than 180°.

Proof. Let's suppose that we are given a triangle $\triangle ABC$ and we want to show that $(\angle A) + (\angle B) < 180°$. First I need to label one more point: choose D so that $D * A * C$. Then

$$(\angle BAC) + (\angle ABC) < (\angle BAC) + (\angle BAD) = 180°.$$



Exterior Angle Th. $(\angle ABC) < (\angle BAD)$

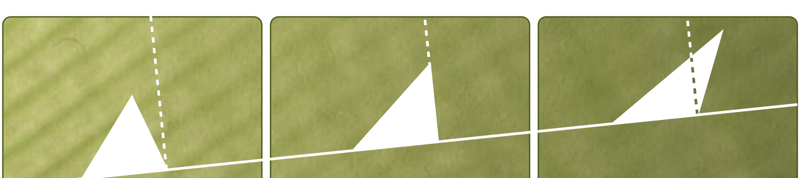
Supplements $\angle BAC$ and $\angle BAD$

□

Note that this means that a triangle cannot support more than one right or obtuse angle– if a triangle has a right angle, or an obtuse angle, then the other two angles have to be acute. That leads to some more terminology.

DEF: ACUTE, RIGHT, AND OBTUSE TRIANGLES

A triangle is *acute* if all three of its angles are acute. A triangle is *right* if it has a right angle. A triangle is *obtuse* if it has an obtuse angle.



Acute *Right* *Obtuse*

The real key to this proof of the Saccheri-Legendre Theorem, the mechanism that makes it work, is the second lemma.

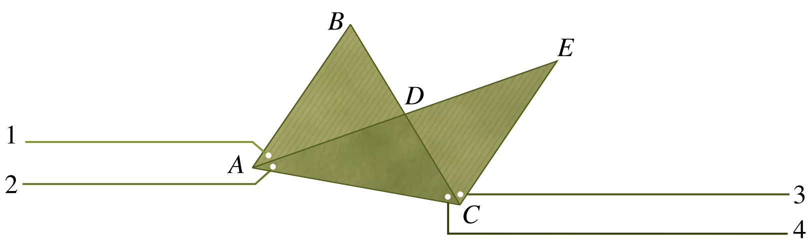
LEMMA TWO
For any triangle $\triangle ABC$, there is another triangle $\triangle A'B'C'$ so that
1. $s(\triangle ABC) = s(\triangle A'B'C')$, and
2. $(\angle A') \leq (\angle A)/2$.

Proof. This is a constructive proof: I am going to describe how to build a triangle from $\triangle ABC$ that meets both of the requirements listed in the theorem. First we are going to need to label a few more points:

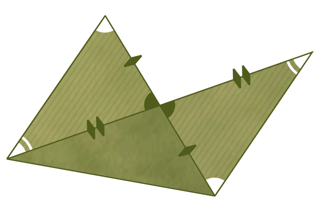
D: the midpoint of BC,
E: on $AD\rightarrow$, so that $A * D * E$ and $AD \simeq DE$.

My claim is that $\triangle ACE$ satisfies both of the conditions (1) and (2). Showing that it does involves comparing angle measures, and with that in mind I think it is helpful to abbreviate some of the angles:

$\angle 1$ for $\angle BAD$, $\angle 2$ for $\angle DAC$, $\angle 3$ for $\angle DCE$, and $\angle 4$ for $\angle ACD$.



The key to showing that $\triangle ACE$ meets requirements (1) and (2) is the pair of congruent triangles formed by carving away the overlap of $\triangle ABC$ and $\triangle ACE$. Notice that by S·A·S



$$\left.\begin{array}{r} BD \simeq CD \\ \angle BDA \simeq \angle CDE \\ DA \simeq DE \end{array}\right\} \longrightarrow \triangle BDA \simeq \triangle CDE \longrightarrow \left\{\begin{array}{l} \angle 1 \simeq \angle E \\ \angle B \simeq \angle 3 \end{array}\right.$$

Condition 1. For the first, all you have to do is compare the two angle sums:

$$s(\triangle ABC) = (\angle A) + (\angle B) + (\angle 4) = (\angle 1) + (\angle 2) + (\angle B) + (\angle 4)$$

$$s(\triangle ACE) = (\angle 2) + (\angle ACE) + (\angle E) = (\angle 2) + (\angle 3) + (\angle 4) + (\angle E).$$

Sure enough, they are the same.

Condition 2. The second part is a little devious, because I can't tell you which angle of $\triangle ACE$ will end up being $\angle A'$. What I can say, though, is that

$$(\angle BAC) = (\angle 1) + (\angle 2) = (\angle E) + (\angle 2).$$

Therefore it isn't possible for both $\angle E$ and $\angle 2$ to measure more than $(\angle BAC)/2$. Let $\angle A'$ be the smaller of the two (or just choose one if they are both the same size). $\square$

Now we can combine those two lemmas into a proof of the Saccheri-Legendre Theorem itself.

Proof. Suppose that there is a triangle $\triangle ABC$ whose angle sum is more than $180°$. In order to keep track of that excess, write

$$s(\triangle ABC) = (180 + x)°.$$

Now let's iterate! According to Lemma 2, there is a triangle

$\triangle A_1B_1C_1$ with the same angle sum but $(\angle A_1) \leq \frac{1}{2}(\angle A)$;

$\triangle A_2B_2C_2$ with the same angle sum but $(\angle A_2) \leq \frac{1}{2}(\angle A_1) \leq \frac{1}{4}(\angle A)$;

$\triangle A_3B_3C_3$ with the same angle sum but $(\angle A_3) \leq \frac{1}{2}(\angle A_2) \leq \frac{1}{8}(\angle A)$;

Starting from an equilateral triangle, the first three iterations.

After going through this procedure n times, we will end up with a triangle $\triangle A_nB_nC_n$ whose angle sum is still $(180+x)^\circ$ but with one very tiny angle– $(\angle A_n) \le \frac{1}{2^n}(\angle A)$. No matter how big $\angle A$ is or how small x is, there is a large enough value of n so that $\frac{1}{2^n}(\angle A) < x$. In that case, the remaining two angles of the triangle $\angle B_n$ and $\angle C_n$ have to add up to more than 180°. According to Lemma 1, this cannot happen. Therefore there cannot be a triangle with an angle sum over 180°. □

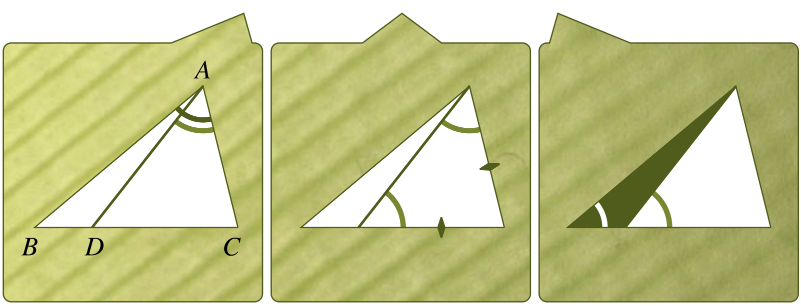
The Scalene Triangle Theorem

The Scalene Triangle Theorem relates the measures of the angles of triangle to the measures of its sides. Essentially, it guarantees that the largest angle is opposite the longest side and that the smallest angle is opposite the shortest side. More precisely

THM: SCALENE TRIANGLE THEOREM
In $\triangle ABC$ suppose that $|BC| > |AC|$. Then $(\angle BAC) > (\angle ABC)$.

Proof. With the results we have established so far, this is an easy one. We need to draw an isosceles triangle into $\triangle ABC$ and that requires one additional point. Since $|BC| > |AC|$, there is a point D between B and C so that $CA \simeq CD$. Then

$$(\angle BAC) > (\angle DAC) = (\angle ADC) > (\angle ABC).$$



D is in the interior of $\angle BAC$. *Isosceles Triangle Th.* *Exterior Angle Th.*

□

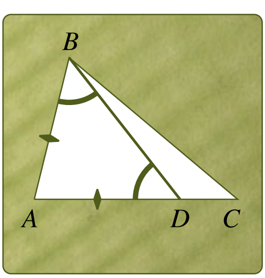
The Triangle Inequality

The Triangle Inequality deals with the lengths of the three sides of a triangle, providing upper and lower bounds for one side in terms of the other two. This is one of the results that has escaped the confines of neutral geometry, though, and you will see triangle inequalities in various disguises is many different areas of math.

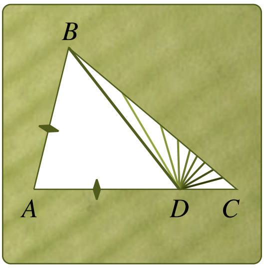
THM: THE TRIANGLE INEQUALITY
In any triangle $\triangle ABC$, the length of side AC is bounded above and below by the lengths of AB and BC:

$$||AB|-|BC|| < |AC| < |AB|+|BC|.$$

Proof. The second inequality is usually what people think of when they think of the Triangle Inequality, and that's the one that I am going to prove. I will leave the proof of the first inequality to you. The second inequality is obviously true if AC isn't the longest side of the triangle, so let's focus our attention on the only really interesting case– when AC is the longest side. As in the proof of the Scalene Triangle Theorem, we are going to build an isosceles triangle inside $\triangle ABC$. To do that, label D between A and C so that $AD \simeq AB$. According to the Isosceles Triangle Theorem, $\angle ADB \simeq \angle ABD$. Thanks to the Saccheri-Legendre Theorem, we now know that these angles can't both be right or obtuse, so they have to be acute. Therefore, $\angle BDC$, which is supplementary to $\angle ADB$, is obtuse. Again, the Saccheri-Legendre Theorem: the triangle $\triangle BDC$ will



In $\triangle ABD$, $\angle B$ and $\angle D$ are congruent, so they must be acute.



In $\triangle BCD$, $\angle D$ is obtuse, so it is the largest angle. Opposite it, BC is the longest side.

only support one obtuse angle, so $\angle BDC$ has to be the largest angle in that triangle. According to the Scalene Triangle Theorem, BC has to be the longest side of $\triangle BDC$. Hence $|DC| < |BC|$. Now let's put it together

$$|AC| = |AD| + |DC| < |AB| + |BC|.$$

$\square$

For proper triangles, the Triangle Inequality promises strict inequalities– $<$ instead of $\leq$. When the three points A, B and C collapse into a straight line, they no longer form a proper triangle, and that is when the inequalities become equalities:

$$\begin{aligned}
&\text{if } C * A * B, \text{ then } |AC| = |BC| - |AB|;\\
&\text{if } A * C * B, \text{ then } |AC| = |AB| - |BC|;\\
&\text{if } A * B * C, \text{ then } |AC| = |AB| + |BC|.
\end{aligned}$$

Exercises

1. Prove the converse of the Scalene Triangle Theorem: in $\triangle ABC$, if $(\angle BAC) > (\angle ABC)$ then $|BC| > |AC|$.

2. Prove the other half of the triangle inequality.

3. Given a triangle $\triangle ABC$, consider the interior and exterior angles at a vertex, say vertex A. Prove that the bisectors of those two angles are perpendicular.

4. Prove that for any point P and line ℓ, there are points on ℓ which are arbitrarily far away from ℓ.

5. Prove that equilateral triangles exist in neutral geometry (that is, describe a construction that will yield an equilateral triangle). Note that all the interior angles of an equilateral triangle will be congruent, but you don't know that the measures of those interior angles is 60°.

6. Prove a strengthened form of the Exterior Angle Theorem: for any triangle, the measure of an exterior angle is greater than or equal to the *sum* of the measures of the two nonadjacent interior angles.

7. Prove that if a triangle is acute, then the line which passes through a vertex and is perpendicular to the opposite side will intersect that side (the segment, that is, not just the line containing the segment).

   Recall that SSA is not a valid triangle congruence theorem. If you know just a little bit more about the triangles in question, though, SSA can be enough to prove triangles congruent. The next questions look at some of those situations.

8. In a right triangle, the side opposite the right angle is called the *hypotenuse*. By the Scalene Triangle Theorem, it is the longest side of the triangle. The other two sides are called the *legs* of the triangle. Consider two right triangles $\triangle ABC$ and $\triangle A'B'C'$ with right angles at C and C', respectively. Suppose in addition that

$$AB \simeq A'B' \quad \& \quad AC \simeq A'C'$$

(the hypotenuses are congruent, as are one set of legs). Prove that $\triangle ABC \simeq \triangle A'B'C'$. This is the H·L congruence theorem for right triangles.

9. Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are *acute* triangles and that

$$AB \simeq A'B' \quad BC \simeq B'C' \quad \angle C \simeq \angle C'.$$

Prove that $\triangle ABC \simeq \triangle A'B'C'$.

10. Consider triangles $\triangle ABC$ and $\triangle A'B'C'$ with

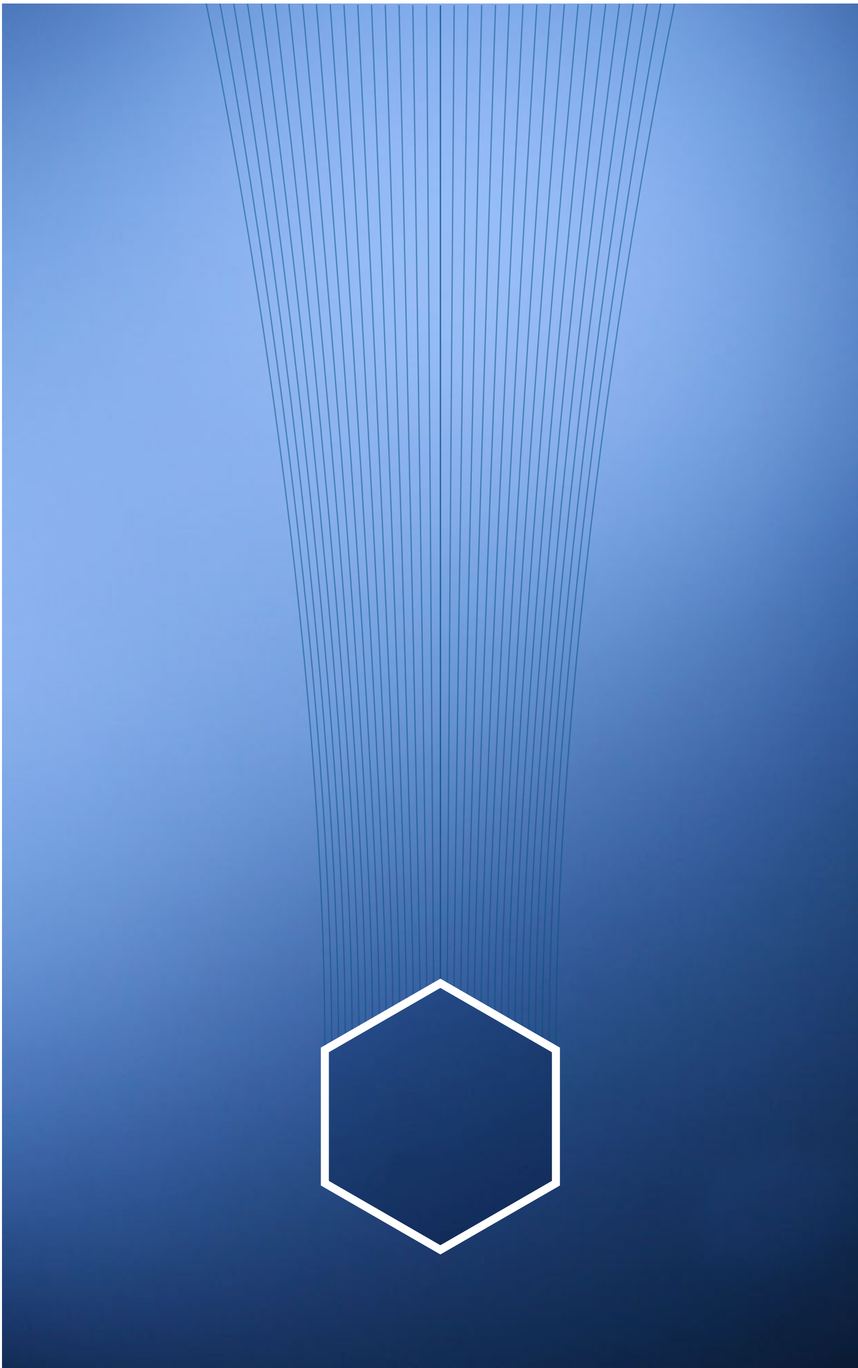
$$AB \simeq A'B' \quad BC \simeq B'C' \quad \angle C \simeq \angle C'.$$

Suppose further that $|AB| > |BC|$. Prove that $\triangle ABC \simeq \triangle A'B'C'$.

References

The proof that I give for the Saccheri-Legendre Theorem is the one I learned from Wallace and West's book [1].

[1] Edward C. Wallace and Stephen F. West. *Roads to Geometry*. Pearson Education, Inc., Upper Saddle River, New Jersey, 3rd edition, 2004.



11 THE MANY SIDES OF **POLYGONS**

We have spent a lot of time talking about triangles, and I certainly do not want to give the impression that we are done with them, but in this lesson I would like to broaden the focus a little bit, and to look at polygons with more than three sides.

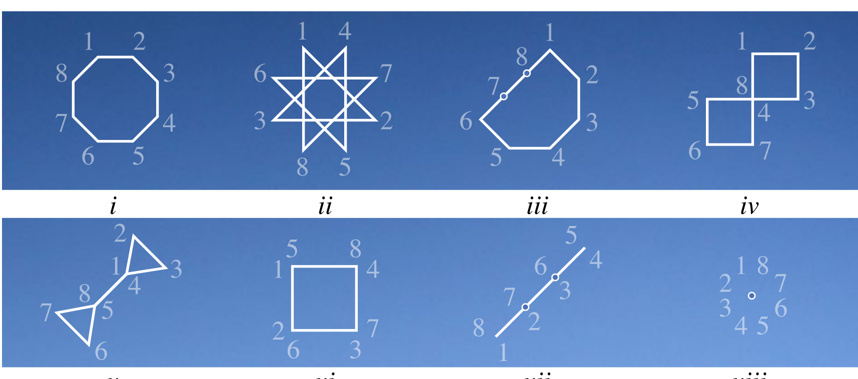
Definitions

Of course the first step is to get a working definition for the term *polygon*. This may not be as straightforward as you think. Remember the definition of a triangle? Three non-colinear points P_1, P_2, and P_3 defined a triangle. The triangle itself consisted of all the points on the segments P_1P_2, P_2P_3, and P_3P_1. At the very least, a definition of a polygon (as we think of them) involves a list of points and segments connecting each point to the next in the list, and then the last point back to the first:

The Vertices: $P_1, P_2, P_3, \ldots, P_n$

The Sides: $P_1P_2, P_2P_3, P_3P_4, \ldots, P_{n-1}P_n, P_nP_1$.

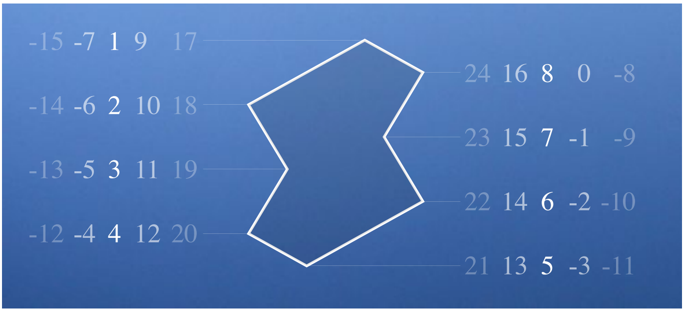
Now the one problem is this– what condition do you want to put on those points? With triangles, we insisted that the three points be non-colinear. What is the appropriate way to extend that beyond $n = 3$? This is not an easy question to answer. To give you an idea of some of the potential issues, let me draw a few configurations of points.



i *ii* *iii* *iv*

v *vi* *vii* *viii*

Which of these do you think should be considered octagons (polygons with eight sides and eight vertices)?



Cyclic labeling of vertices

While you are mulling over that question, let me distract you by talking about notation. No matter what definition of polygon you end up using, your vertices will cycle around: $P_1, P_2, \ldots, P_n$ and then back to the start P_1. Because polygons do loop back around like this, sometimes you end up crossing from P_n back to P_1. For example, look at the listing of the sides of the polygon– all but one of them can be written in the form P_iP_{i+1}, but the last side, P_nP_1, doesn't fit that pattern. A proof involving the sides would have to go out of its way to be sure to mention that last side, and that is just not going to be very elegant. After all, other than the notation, the last side is not any different from the previous sides– it really should not need its own case. Fortunately, there is an easy way to sidestep this issue. What we can do is make our subscripts cycle just like the points do. Rather than using integer subscripts for the vertices, use integers modulo n (where n is the number of vertices). That way, for instance, in a polygon with eight vertices, P_9 and P_1 would stand for the same point since $9 \cong 1 \mod 8$, and the sides of the polygon would be P_iP_{i+1} for $1 \leq i \leq 8$.

Now let's get back to the question of a definition. As I said at the start of the lesson, I think that there is still a spectrum of opinion on how a polygon should be defined. Some geometers (such as Grünbaum in *Are your polyhedra the same as my polyhedra* [2]) will tell you that any ordered listing of n points should define a polygon with n vertices and n sides. This includes listings where some or even all points are colinear or coinciding and can therefore can lead to some unexpected configurations: a six-sided polygon that appears to have only three sides, a triangle that looks like a line segment, a four-sided polygon that looks like a point. If you can get past the initial strangeness, though, there is definitely something to be said for this all-inclusive approach: for one thing, you never have to worry that moving points around would cause (for instance) your four-sided polygon to no longer be a four-sided polygon. This liberal definition would go something like this:

DEF: POLYGON (INCLUSIVE VERSION)
Any ordered list of points $\{P_i | 1 \leq i \leq n\}$ defines a polygon, written $P_1P_2 \cdots P_n$, with vertices P_i, $1 \leq i \leq n$, and sides P_iP_{i+1}, $1 \leq i \leq n$.

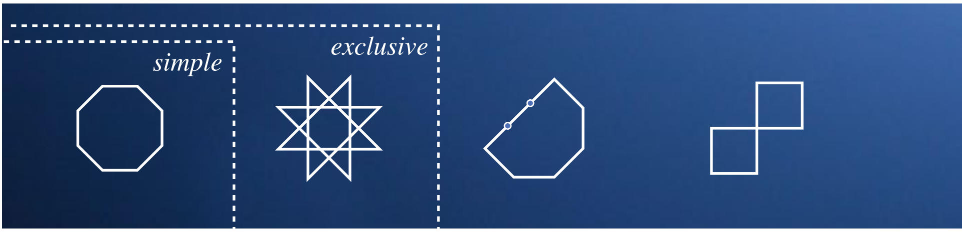
Other geometers like to put a few more restrictions on their polygons. I suspect that the most common objections to this all-inclusive definition would be:

(1) This collapsing of the vertices down to a single point or a single line as shown in illustrations (vii) and (viii) is unacceptable– polygons should have a two-dimensionality to them.

(2) The edges of a polygon should not trace back over one another as shown in illustrations (v) and (vi)– at most two edges should intersect each other once.

(3) On the topic of intersecting edges, only consecutive edges should meet at a vertex. Configurations such as the one shown in illustration (iv) do not define a single polygon, but rather several polygons joined together.

I don't know to what extent these added restrictions are historical conventions and to what extent they are truly fundamental to proving results on polygons. Let me point out though, that this all-inclusive definition doesn't quite work with our previous definition of a triangle: three colinear points would define a three-sided polygon, but not a triangle. Somehow, that just does not seem right. Were we to now to go back and liberalize our definition of a triangle to include these remaining three-sided polygons, it would cost us some theorems. For instance, neither A·S·A nor A·A·S would work in the case when all three vertices are colinear. So for that reason, let me also give a more restrictive definition of polygon that addresses the three concerns listed above.



i *ii* *iii* *iv*

DEF: POLYGON (EXCLUSIVE VERSION)
Any ordered list of points $\{P_i | 1 \leq i \leq n\}$ which satisfies the conditions

(1) no three consecutive points P_i, P_{i+1}, and P_{i+2} are colinear;
(2) if $i \neq j$, then P_iP_{i+1} and P_jP_{j+1} share at most one point;
(3) if $P_i = P_j$, then $i = j$;

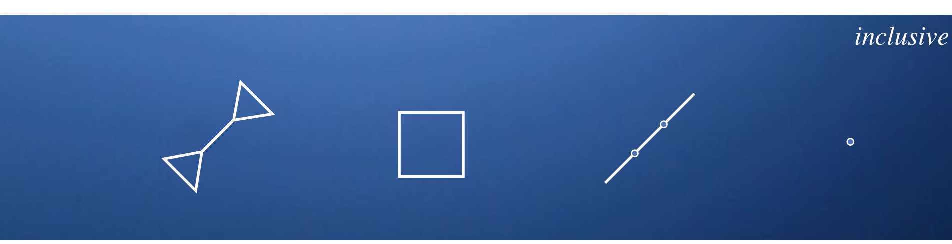
defines a polygon, written $P_1P_2 \cdots P_n$, with vertices P_i, $1 \leq i \leq n$, and sides P_iP_{i+1}, $1 \leq i \leq n$.

The crux of it is this: too liberal a definition and you are going to have to make exceptions and exclude degenerate cases; too conservative a definition and you end up short-changing your results by not expressing them at their fullest generality. After all of that, though, I have to say that I'm just not that worried about it, because for the most part, the polygons that we usually study are more specialized than either of those definitions– they are what are called *simple polygons*. You see, even in the more "exclusive" definition, the segments of a polygon are permitted to criss-cross one another. In a simple polygon, that type of behavior is not tolerated.

DEF: SIMPLE POLYGON
Any ordered list of points $\{P_i | 1 \leq i \leq n\}$ which satisfies the conditions

(1) no three consecutive points P_i, P_{i+1}, and P_{i+2} are colinear;
(2) if $i \neq j$ and P_iP_{i+1} intersects P_jP_{j+1} then either $i = j+1$ and the intersection is at $P_i = P_{j+1}$ or $j = i+1$ and the intersection is at $P_{i+1} = P_j$;

defines a simple polygon, written $P_1P_2 \cdots P_n$, with vertices P_i, $1 \leq i \leq n$, and sides P_iP_{i+1}, $1 \leq i \leq n$.

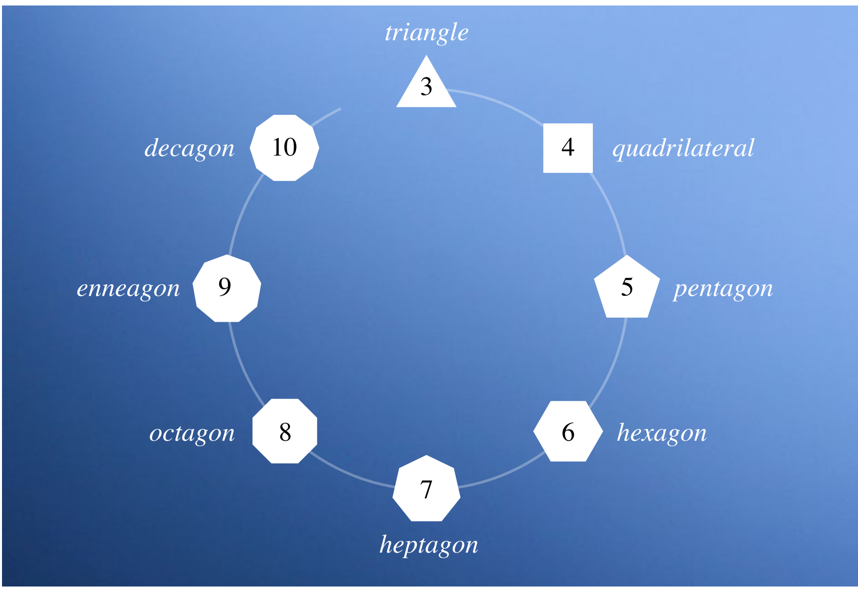


v *vi* *vii* *viii*

No matter how you choose to define a polygon, the definition of one important invariant of a polygon does not change:

DEF: PERIMETER
The perimeter P of a polygon is the sum of the lengths of its sides:

$$P = \sum_{i=1}^{n} |P_i P_{i+1}|.$$



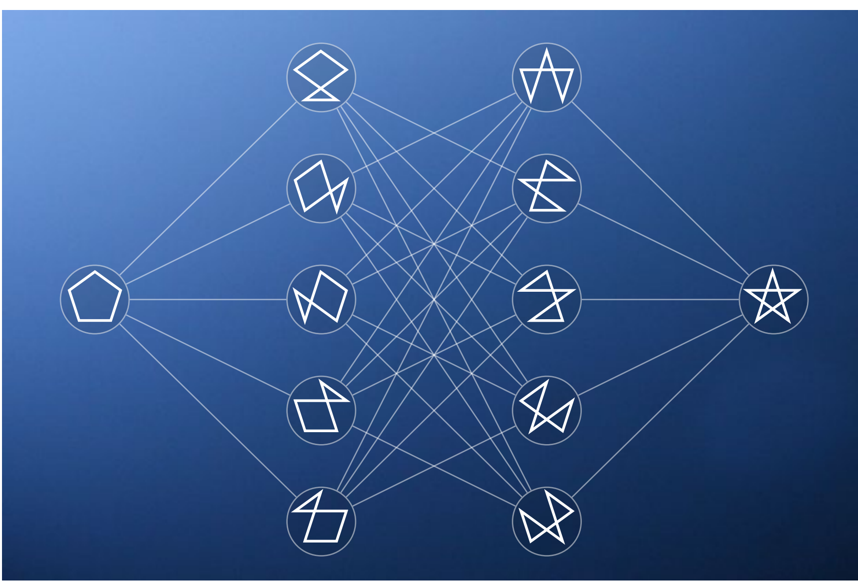
Names of polygons based upon the number of sides (and vertices).

Counting polygons

Two polygons are the same if they have the same vertices *and* the same edges. That means that the order that you list the vertices generally *does* matter– different orders can lead to different sets of sides. Not all rearrangements of the list lead to new polygons though. For instance, the listings $P_1P_2P_3P_4$ and $P_3P_4P_1P_2$ and $P_4P_3P_2P_1$ all define the same polygon: one with sides P_1P_2, P_2P_3, P_3P_4 and P_4P_1. More generally, any two listings which differ either by a cycling of the vertices or by a reversal of the order of one of those cyclings will describe the same polygon.

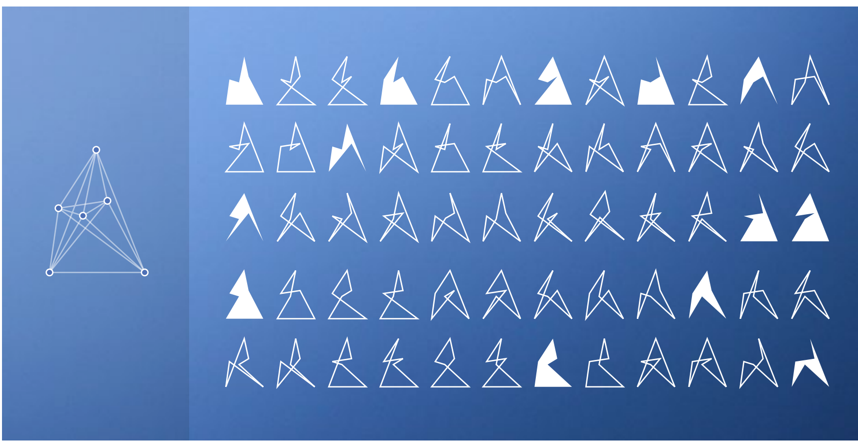
The 24 permutations of $1, 2, 3, 4$ *and the corresponding polygons on four points.*

So how many possible polygons are there on n points? That depends upon what definition of polygon you are using. The most inclusive definition of polygon leads to the easiest calculation, for in that case, any configuration on n points results in a polygon. As you probably know from either probability or group theory, there are $n!$ possible orderings of n distinct elements. However for each such list there are n cyclings of the entries and n reversals of those cyclings, leading to a total of $2n$ listings which all correspond to the same polygon. Therefore, there are $n!/(2n) = (n-1)!/2$ possible polygons that can be built on n vertices. Notice that when $n = 3$, there is only one possibility, and that is why none of this was an issue when we were dealing with triangles.



The 12 polygons on a configuration of five points. In this illustration, segments connect two polygons which differ by a swap of two adjacent vertices.

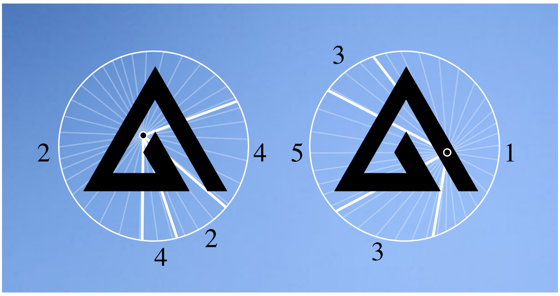
If instead you are using the more exclusive definition of a polygon, then things are a bit more complicated. If the vertices are in "general position" so that any combination of segments P_iP_j satisfies the requirements outlined in that definition, then there are just as many exclusive polygons as inclusive polygons: $(n-1)!/2$. Probabilistically, it is most likely that any n points will be in such a general position, but it is also true that as n grows, the number of conditions required to attain this general position increases quite rapidly. Even less understood is the situation for simple polygons. The condition of simplicity throws the problem from the relatively comfortable world of combinatorics into a much murkier geometric realm.



Thirteen of the sixty polygons on this configuration of six points are simple.

Interiors and exteriors

One characteristic of the triangle is that it chops the rest of the plane into two sets, an interior and an exterior. It isn't so clear how to do that with a polygon (this is particularly true if you are using the inclusive definition of the tem, but to a lesser extent is still true with the exclusive definition). Simple polygons, though, do separate the plane into interior and exterior. This is in fact a special case of the celebrated Jordan Curve Theorem, which states that every simple closed curve in the plane separates the plane into an interior and an exterior. The Jordan Curve Theorem is one of those notorious results that seems like you could knock out in an afternoon, but is actually brutally difficult. In the special case of simple polygons, our case, there are simpler proofs. I am going to describe the idea behind one such proof from *What is Mathematics?* by Courant and Robbins [1].



Rays from a point. The number of intersections with a polygon (in black) depends upon which ray is chosen, but the parity (even or odd) does not.

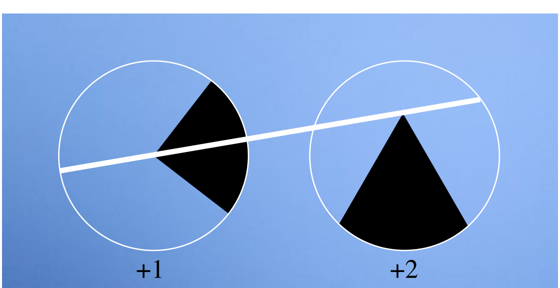
THM: POLYGONAL PLANE SEPARATION

Every simple polygon separates the remaining points of the plane into two connected regions.

Proof. Let $\mathscr{P}$ be a simple polygon, and let p be a point which is not on $\mathscr{P}$. Now let's look at a ray R_p whose endpoint is p. As long as R_p does not run exactly along an edge, it will intersect the edges of $\mathscr{P}$ a finite number of times (perhaps none). You want to think of each such intersection as a crossing of R_p into or out of $\mathscr{P}$.

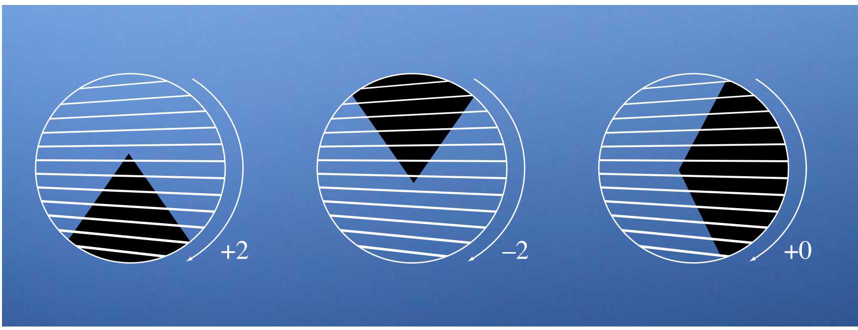
Since there are only finitely many intersections, they are all within a finite distance of P. That means that eventually R_p will pass beyond all the points of $\mathscr{P}$. This is the essence of this argument: eventually the ray is outside of the polygon, so by counting back the intersections crossing into and out of the polygon, we can figure out whether the beginning of the ray, P is inside or outside of $\mathscr{P}$. The one situation where we have to be a little careful is when R_p intersects a vertex of $\mathscr{P}$. Here is the way to count those intersections:

$$\begin{cases} \text{once if } R_p \text{ separates the two neighboring edges;} \\ \text{twice if } R_p \text{ does not separate them.} \end{cases}$$



Procedure for counting intersections at a vertex.

Now when you count intesections this way, the number of intersections depends not just upon the point p, but also upon the of direction of R_p. The key, though, is that there is one thing which does not depend upon the direction– whether the number of intersections is odd or even, the "parity" of p. To see why, you have to look at what happens as you move the ray R_p around, and in particular what causes the number of intersections to change. Without giving overly detailed explanation, changes can

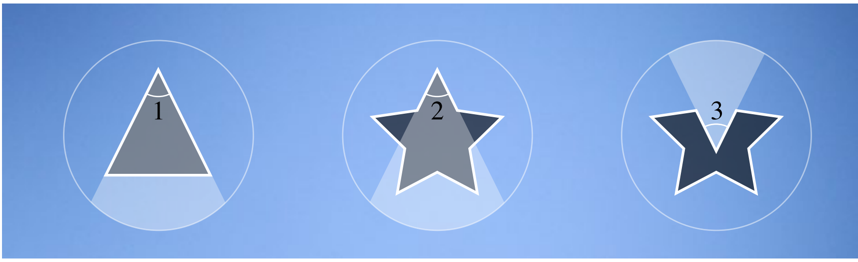


As the ray shifts across a vertex, the intersection count changes by +2, -2, *or* 0*, all even numbers.*

only happen when R_p crosses one of the vertices of $\mathscr{P}$. Each such vertex crossing corresponds to either an increase in 2 in the number of crossings, a decrease by 2 in the number of crossings, or no change in the number of crossings. In each case, the parity is not changed. Therefore $\mathscr{P}$ separates the remaining points of the plane into two sets– those with even parity and those with odd parity. Furthermore, each of those sets is connected in the sense that by tracing just to one side of the edges of $\mathscr{P}$, it is possible to lay out a path of line segments connecting any two points with even parity, or any two points with odd parity. □

DEF: POLYGON INTERIOR AND EXTERIOR
For any simple polygon $\mathscr{P}$, the set of points with odd parity (as described in the last proof) is called the *interior* of $\mathscr{P}$. The set of points with even parity is called the *exterior* of $\mathscr{P}$.

I will leave it to you to prove the intuitively clear result: that a polygon's interior is always a bounded region and that its exterior is always an unbounded region.

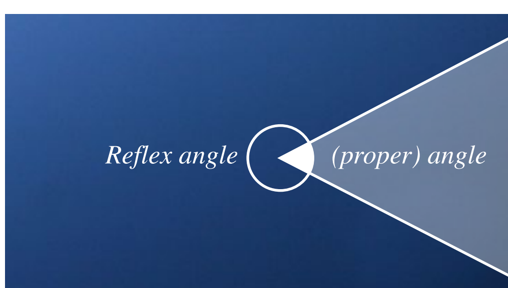


Angle interiors and polygon interiors.

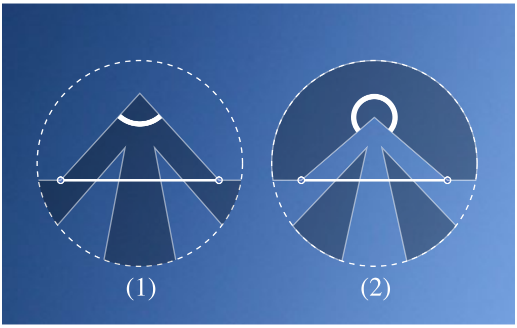
Interior angles: two dilemmas

Now I want to talk a little bit about the interior angles of a simple polygon. If you would, please look at the three marked angles in the polygons above. The first, $\angle 1$ is the interior angle of a triangle. You can see that the entire interior of the triangle is contained in the interior of the angle, and that seems proper, that close connection between the interiors of the interior angles and the interior of the triangle. Now look at $\angle 2$, and you can see that for a general simple polygon, things do not work quite as well: the entire polygon does not lie in the interior of this angle. But at least the part of the polygon interior which is closest to that vertex is in the interior of that angle. Finally look at $\angle 3$: the interior of $\angle 3$ encompasses exactly none of the interior of the polygon– it is actually pointing away from the polygon.

Let me address the issue surrounding $\angle 3$ first. We have said that two non-opposite rays define a single angle, and later established a measure for that angle– some number between 0 and 180°. But really, two rays like this divide the plane into *two* regions, and correspondingly, they should form *two* angles. One is the proper angle which we have already dealt with. The other angle is what is called a *reflex angle*. Together, the measures of the proper angle and the reflex angle formed by any two rays should add up to



A pair of explementary angles.



The dark region shows the polygon interior around a vertex. In (1), the connecting segment begins in the interior, so the interior angle is the proper angle. In (2), the connecting segment begins in the exterior, so the interior angle is the reflex angle.

$360°$. There does not seem to be a standard bit of terminology to describe this relationship between angles; I have seen the term "conjugate" as well as the term "explementary". So the problem with $\angle 3$ is that the interior angle isn't the proper angle, but instead, that it is its explement.

Now as long as the polygon is fairly simple (no pun intended) this is all fairly clear, but suppose that we were looking at an angle $\angle P_i$ in a much more elaborate polygon. Should the interior angle at P_i be the proper angle $\angle P_{i-1}P_iP_{i+1}$ or its conjugate? Well, to answer that question, you need to look at the segment $P_{i-1}P_{i+1}$. It may cross into and out of the interior of the polygon, but if the interior angle is the proper angle, then the first and last points of $P_{i-1}P_{i+1}$ (the ones closest to P_{i-1} and P_{i+1}) will be in the interior of the polygon. If the interior angle is the reflex angle, then the first and last points of $P_{i-1}P_{i+1}$ won't be in the interior of the polygon.

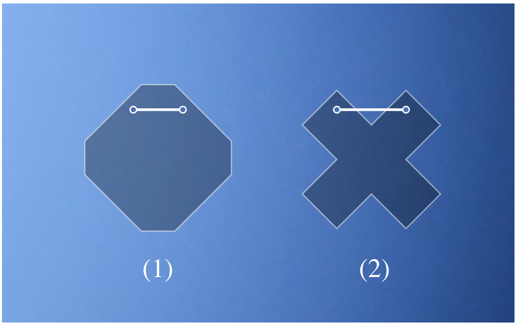
With the interior angles of a polygon now properly accounted for, we can define what it means for two polygons to be congruent.

DEF: POLYGON CONGRUENCE

Two polygons $\mathscr{P} = P_1P_2\cdots P_n$ and $\mathscr{Q} = Q_1Q_2\cdots Q_n$ are congruent, written $\mathscr{P} \simeq \mathscr{Q}$ if all corresponding sides and interior angles are congruent:

$$P_iP_{i+1} \simeq Q_iQ_{i+1} \quad \& \quad \angle P_i \simeq \angle Q_i, \text{ for all } i.$$

Now let's take a look at $\angle 2$, where not all of the interior of the polygon lies in the interior of the angle. The problem here is a little bit more intrinsic– I don't think you are going to be able to get around this one by fiddling with definitions (well, not at least without making a lot of questionable compromises). There is, though, a class of simple polygon for which the polygon interior always lies in the interior of each interior angle. These are the convex polygons.



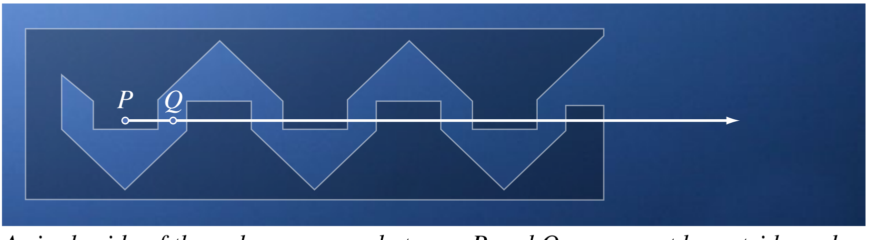
A (1) convex and (2) a non-convex polygon. In the second, a segment joins two points in the interior, but passes outside of the polygon.

DEF: CONVEX POLYGON
A polygon $\mathscr{P}$ is *convex* if, for any two points p and q in the interior of $\mathscr{P}$, the entire line segment pq is in the interior of $\mathscr{P}$.

Convexity is a big word in geometry and it comes up in a wide variety of contexts. Our treatment here will be very elementary, and just touch on the most basic properties of a convex polygon.

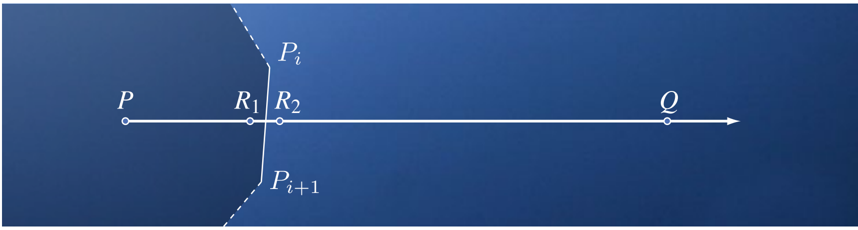
THM: CONVEXITY 1
If $\mathscr{P} = P_1P_2 \cdots P_n$ is a convex polygon, then all the points of the interior of $\mathscr{P}$ lie on the same side of each of the lines P_iP_{i+1}.

Proof. The fundamental mechanism that makes this proof work is the way that we defined the interior and exterior of a polygon by drawing a ray out and counting how many times it intersects the sides of $\mathscr{P}$. Suppose that P and Q lie on opposite sides of a segment P_iP_{i+1}, so that PQ intersects P_iP_{i+1}. Suppose further that PQ intersects no other sides of the polygon. Then the ray $PQ\rightarrow$ will intersect $\mathscr{P}$ one more time than the ray $(QP\rightarrow)^{op}$. Therefore P and Q will have different parities, and so one of P and Q will be an interior point and the other an exterior point.



A single side of the polygon comes between P and Q– one must be outside and one must be inside.

Now on to the proof, a proof by contradiction. Suppose that both P and Q are in the interior of a convex polygon, but that they are on the opposite sides of $\leftarrow P_iP_{i+1}\rightarrow$. After the previous discussion, it is tempting to draw a picture that looks like

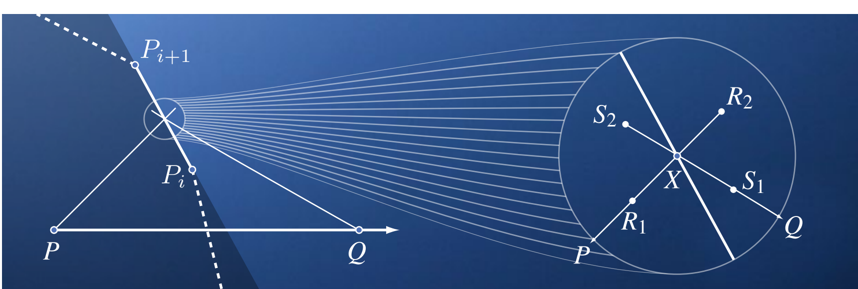


In that case, only one of R_1, R_2 can be in the interior of $\mathscr{P}$ and so $\mathscr{P}$ can't be convex and we have our contradiction. But that misses an important (and indeed likely) scenario– the one in which PQ intersects the line $\leftarrow P_iP_{i+1}\rightarrow$ but not the segment P_iP_{i+1}. To deal with that scenario, we are going to have to maneuver the intersection so that it does occur on the segment, which requires a bit more delicate argument.

Choose a point X which is between P_i and P_{i+1}. We will relay the interior/exterior information from P and Q back to points which are in close proximity to X. Choose points R_1, R_2, S_1 and S_2 so that

$$P * R_1 * X * R_2 \quad Q * S_1 * X * S_2.$$

In addition, we want to make sure that these points are so close together that none of the other sides of $\mathscr{P}$ get in the way, so we will require (1) R_1S_1 does intersect the side P_iP_{i+1}, but that (2) none of the edges other than P_iP_{i+1} comes between any two of these points. A polygon only has finitely many edges, so yes, it is possible to do this. Then R_1 and R_2 lie on different sides of the segment P_iP_{i+1}, so one is in the interior and one



is in the exterior. Suppose that R_2 is the interior point. Then, since $\mathscr{P}$ is convex, and R_1 is between two interior points P and R_2, R_1 must also be an interior point. Since R_1 and R_2 cannot both be interior points, that means that R_1 is the interior point. Applying a similar argument to Q, S_1 and S_2, you can show that S_1 must also be an interior point. But now R_1 and S_1 are on opposite sides of P_iP_{i+1}, so they cannot both be interior points. This is the contradiction. □

There are a couple immediate corollaries of this– I am going to leave the proofs of both of these to you.

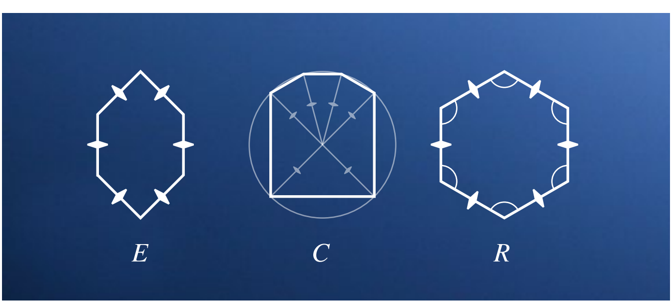
THM: CONVEXITY 2
If $\mathscr{P}$ is a convex polygon, then the interior of $\mathscr{P}$ lies in the interior of each interior angle $\angle P_i$.

THM: CONVEXITY 3
If $\mathscr{P}$ is a convex polygon, then each of its interior angles is a proper angle, not a reflex angle.

Polygons of note

To finish this chapter, I want to mention a few particularly well-behaved types of polygons.

TYPES OF POLYGONS
An *equilateral* polygon is one in which all sides are congruent. A *cyclic* polygon is one in which all vertices are equidistant from a fixed point (hence, all vertices lie on a circle, to be discussed later). A *regular* polygon is one in which all sides are congruent and all angles are congruent.

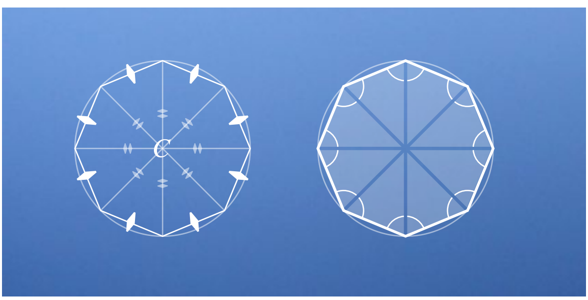


E: equilateral
C: cyclic
R: regular

The third of these types is actually a combination of the previous two types as the next theorem shows.

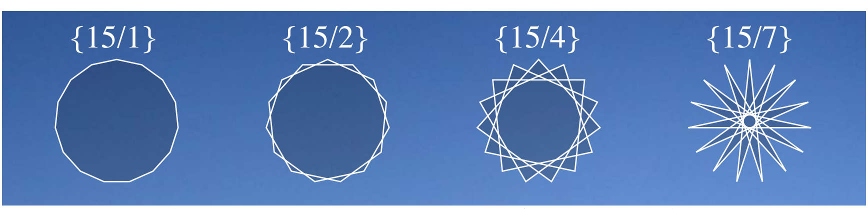
THM: EQUILATERAL + CYCLIC
A polygon $\mathscr{P}$ which is both equilateral and cyclic is regular.

Proof. We need to show that the interior angles of $\mathscr{P}$ are all congruent. Let C be the point which is equidistant from all points of $\mathscr{P}$. Divide $\mathscr{P}$ into a set of triangles by constructing segments from each vertex to C. For any of these triangles, we wish to distinguish the angle at C, the central angle, from the other two angles in the triangle. Note that the two constructed sides of these triangles are congruent. By the Isosceles Triangle Theorem, the two non-central angles are congruent. As well, by S·S·S, all of these triangles are congruent to each other. In particular, all non-central angles of all the triangles are congruent. Since adjacent pairs of such angles comprise an interior angle of $\mathscr{P}$, the interior angles of $\mathscr{P}$ are congruent. □



Because of S-S-S and the Isosceles Triangle Theorem, polygons which are equilateral and cyclic are regular.

While we normally think of regular polygons as I have shown them above, there is nothing in the definition that requires a regular polygon to be simple. In fact, there are non-simple regular polygons– such a polygon is called a *star polygon*.



There is a regular star n-gon for each integer p between 1 *and n/*2 *that is relatively prime to n. Shown here: n=*15*. The* {*n/p*} *notation is called the Schläfli symbol.*

Exercises

1. Verify that a triangle is a convex polygon.

2. A diagonal of a polygon is a segment connecting nonadjacent vertices. How many diagonals does an n-gon have?

3. Prove theorems 2 and 3 on convexity.

4. Prove that a regular convex polygon is cyclic (to find that equidistant point, you may have to consider the odd and even cases separately).

5. Prove that if a polygon is convex, then all of its diagonals lie entirely in the interior of the polygon (except for the endpoints).

6. Prove that if a polygon is not convex, then at least one of its diagonals does not lie entirely in the interior of the polygon.

7. Verify that the perimeter of any polygon is more than twice the length of its longest side.

8. Prove that the sum of the interior angles of a convex n-gon is at most $180°(n-2)$.

9. Prove that if a polygon $\mathscr{P}$ is convex, then there are no other simple polygons on that configuration of vertices.

References

[1] Richard Courant and Herbert Robbins. *What is Mathematics? : an elementary approach to ideas and methods*. Oxford University Press, London, 1st edition, 1941.

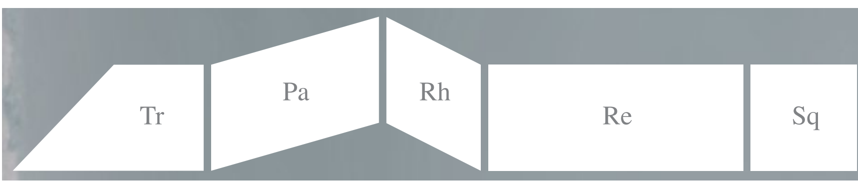
[2] Branko Grünbaum. Are your polyhedra the same as my polyhedra? *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, 2003.

12 FIVE EASY PIECES
QUADRILATERAL CONGRUENCE THEOREMS

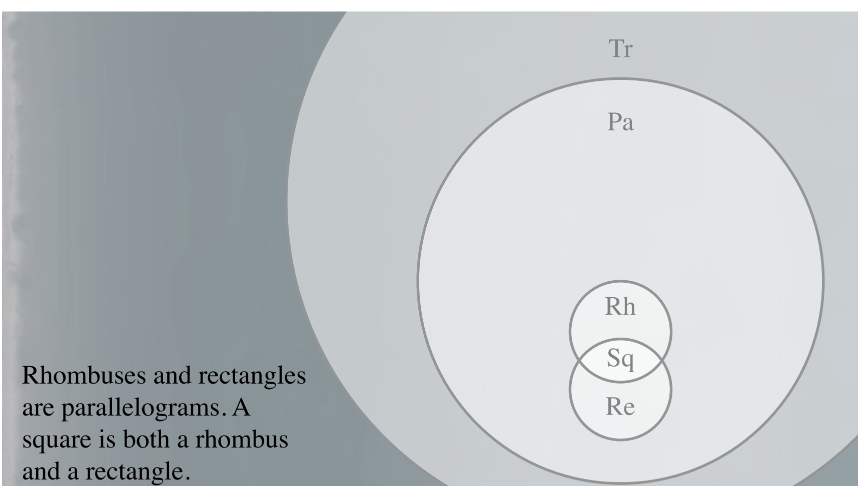
This is the last lesson in neutral geometry. After this, we will allow ourselves one more axiom dealing with parallel lines, and that is the axiom which turns neutral geometry into Euclidean geometry. Before turning down the Euclidean path, let's spend just a little time looking at quadrilaterals. The primary goal of this section will be to develop quadrilateral congruence theorems similar to the triangle congruence theorems we picked up in earlier lessons.

Terminology

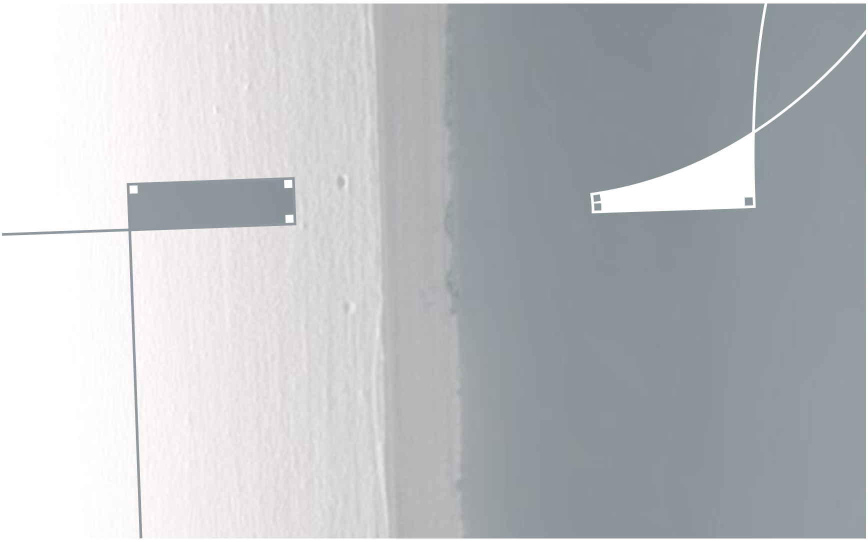
Before I start working on congruence theorems, though, let me quickly run through the definitions of a few particular types of quadrilaterals.



Trapezoid	a pair of parallel sides
Parallelogram	two pairs of parallel sides
Rhombus	four congruent sides
Rectangle	four right angles
Square	four congruent sides and four right angles



Rhombuses and rectangles are parallelograms. A square is both a rhombus and a rectangle.



Quadrilaterals with three right angles. On the left, in Euclidean geometry, the fourth angle is a right angle. On the right, in non-Euclidean geometry, the fourth angle is acute.

One of the risks that you run when you define an object by requiring it to have certain properties, as I have done above, is that you may define something that cannot be– something like an equation with no solution. The objects I have defined above are all such common shapes in everyday life that we usually don't question their existence. Here's the interesting thing though– in neutral geometry, there is no construction which guarantees you can make a quadrilateral with four right angles– that is, neutral geometry does not guarantee the existence of rectangles or squares. At the same time, it does nothing to prohibit the existence of squares or rectangles either. You can make a quadrilateral with three right angles pretty easily, but once you have done that, you have no control over the fourth angle, and the axioms of neutral geometry are just not sufficient to prove definitively whether or not that fourth angle is a right angle. This is one of the fundamental differences that separates Euclidean geometry from non-Euclidean geometry. In Euclidean geometry, the fourth angle is a right angle, so there are rectangles. In non-Euclidean geometry, the fourth angle cannot be a right angle, so there are no rectangles. When we eventually turn our attention to non-Euclidean geometry, I want to come back to this– I would like to begin that study with a more thorough investigation of these quadrilaterals that try to be like rectangles, but fail.

Quadrilateral Congruence

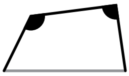
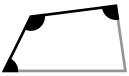
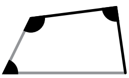
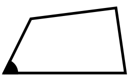
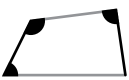
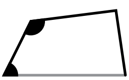
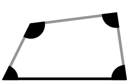
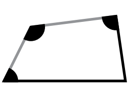
I feel that many authors view the quadrilateral congruences as a means to an end, and as such, tend to take a somewhat ad hoc approach to them. I think I understand this approach– the quadrilateral congruence theorems themselves are a bit bland compared to their application. Still, I want to be a bit more systematic in my presentation of them. In the last chapter we looked at several classes of polygons. To recap:

$$\{\text{convex polygons}\} \subset \{\text{simple polygons}\} \subset \{\text{polygons}\}.$$

For what we are going to be doing in this book, we really only need the congruence results for convex quadrilaterals, but I am going to try to tackle the slightly broader question of congruence for simple quadrilaterals. While the even broader question of congruence for non-simple quadrilaterals would be interesting, I think it is just too far of a detour.

By definition, two quadrilaterals are congruent if four corresponding sides and four corresponding interior angles are congruent– that's a total of eight congruences. Each congruence theorem says that you can guarantee congruence with some subset of that list. If you recall, for triangles you generally needed to know three of the six pieces of information. For quadrilaterals, it seems that the magic number is five. So what I would like to do in this lesson is to look at all the different possible combinations of five pieces (sides and angles) of a quadrilateral and determine which lead to valid congruence theorems. I won't give all the proofs or all the counterexamples (that way you can tackle some of them on your own), but I will provide the framework for a complete classification.

The first step is some basic combinatorics. Each of these theorems has a five letter name consisting of some mix of *S*s and *A*s. When forming this name, there are two choices, *S* and *A* for each of the five letters, and so there are a total of $2^5 = 32$ possible names. Two of these, S·S·S·S·S and A·A·A·A·A, don't make any sense in the context of quadrilateral congruences, though, since a quadrilateral doesn't have five sides or five angles. That leaves thirty different words. Now it is important to notice that not all of these words represent fundamentally different information about the quadrilaterals themselves. For instance, S·S·A·S·A and A·S·A·S·S both represent the same information, just listed in reverse order. Similarly, S·S·A·S·S and S·S·S·S·A both represent the same information– four sides and one angle. Once those equivalences are taken into consideration, we are left with ten potential quadrilateral congruences.

	Word	**Variations**		**Valid congruence?**
	S·A·S·A·S			yes
	A·S·A·S·A			yes
	A·A·S·A·S	SASAA		yes
	S·S·S·S·A	SSSAS SASSS	SSASS ASSSS	no$^{(*)}$
	A·S·A·A·S	SAASA		no
	A·S·A·S·S	SSASA		no
	A·S·S·A·S	SASSA		no
	A·A·A·A·S	AAASA ASAAA	AASAA SAAAA	no
	S·S·S·A·A	AASSS SAASS	ASSSA SSAAS	no
	A·A·A·S·S	SAAAS ASSAA	SSAAA AASSA	yes

$^{(*)}$ a valid congruence theorem for *convex* quadrilaterals

Table 1. Quadrilateral congruence theorems.

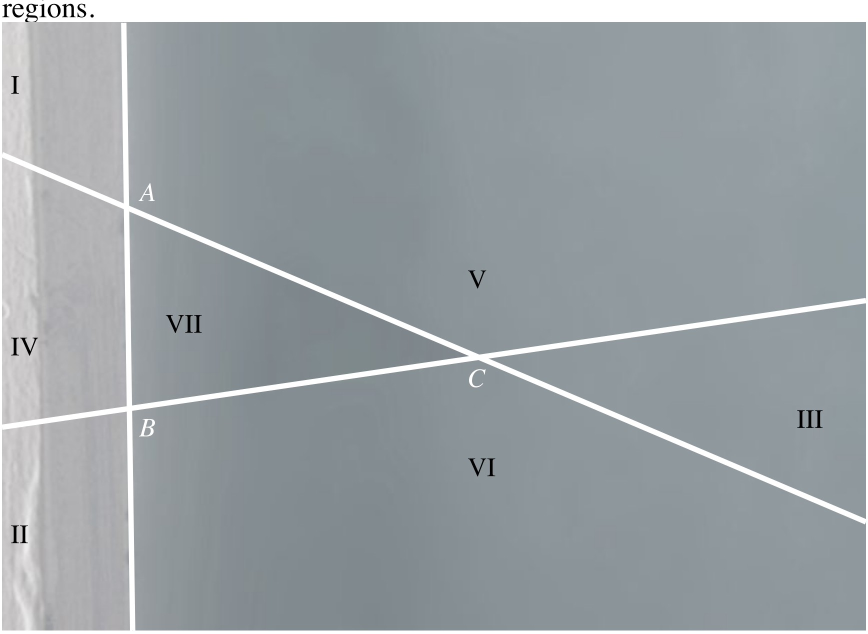
S·A·S·A·S, A·S·A·S·A, and A·A·S·A·S

Each of these is a valid congruence theorem for simple quadrilaterals. The basic strategy for their proofs is to use a diagonal of the quadrilateral to separate it into two triangles, and then to use the triangle congruence theorems. Now the fact that I am allowing both convex and non-convex quadrilaterals in this discussion complicates things a little bit, so let's start by examining the nature of the diagonals of a quadrilateral. Yes, I will be leaving out a few details here (more than a few to be honest) so you should feel free to work out any tricky details for yourself.

Consider a quadrilateral $\square ABCD$ (I am going to use a square symbol to denote a simple quadrilateral). What I want to do is to look at the position of the point D relative to the triangle $\triangle ABC$. Each of the three lines $\leftarrow AB \rightarrow$, $\leftarrow BC \rightarrow$, and $\leftarrow AC \rightarrow$ separate the plane into two pieces. It is not possible, though, for any point of the plane to simultaneously be

(1) on the opposite side of AB from C
(2) on the opposite side of AC from B, and
(3) on the opposite side of BC from A.

Therefore the lines of $\triangle ABC$ divide the plane into seven $(2^3 - 1)$ distinct regions.



The seven "sides" of a triangle.

D is in region	is $\square ABCD$ simple?	D is on the same side of : BC as A	AC as B	AB as C	reflex angle	interior diagonal: AC	BD
I	✓	✓			A	✓	
II	✓		✓		B		✓
III	✓			✓	C	✓	
IV		✓	✓		–	–	–
V	✓	✓		✓	none	✓	✓
VI			✓	✓	–	–	–
VII	✓	✓	✓	✓	D		✓

Table 2. The diagonals of a quadrilateral

Now for each of these seven regions, we can determine whether the diagonals AC and BD are in the interior of $\square ABCD$. Let me point out that this is always an all-or-nothing proposition– either the entire diagonal lies in the interior (excepting of course the endpoints) or none of it does. Additionally, in each case, a diagonal lies in the interior of a quadrilateral if and only if it lies in the interior of both the angles formed by $\square ABCD$ at its endpoints. What I mean is that if, for example, AC is in the interior of $\square ABCD$, then AC will be in the interior of both $\angle DAB$ and $\angle BCD$. If AC isn't in the interior of $\square ABCD$, then AC will not be in the interior of either $\angle DAB$ or $\angle BCD$.

With the diagonals now properly sorted, we can address the congruence theorems directly. Perhaps the most useful of them all is S·A·S·A·S.

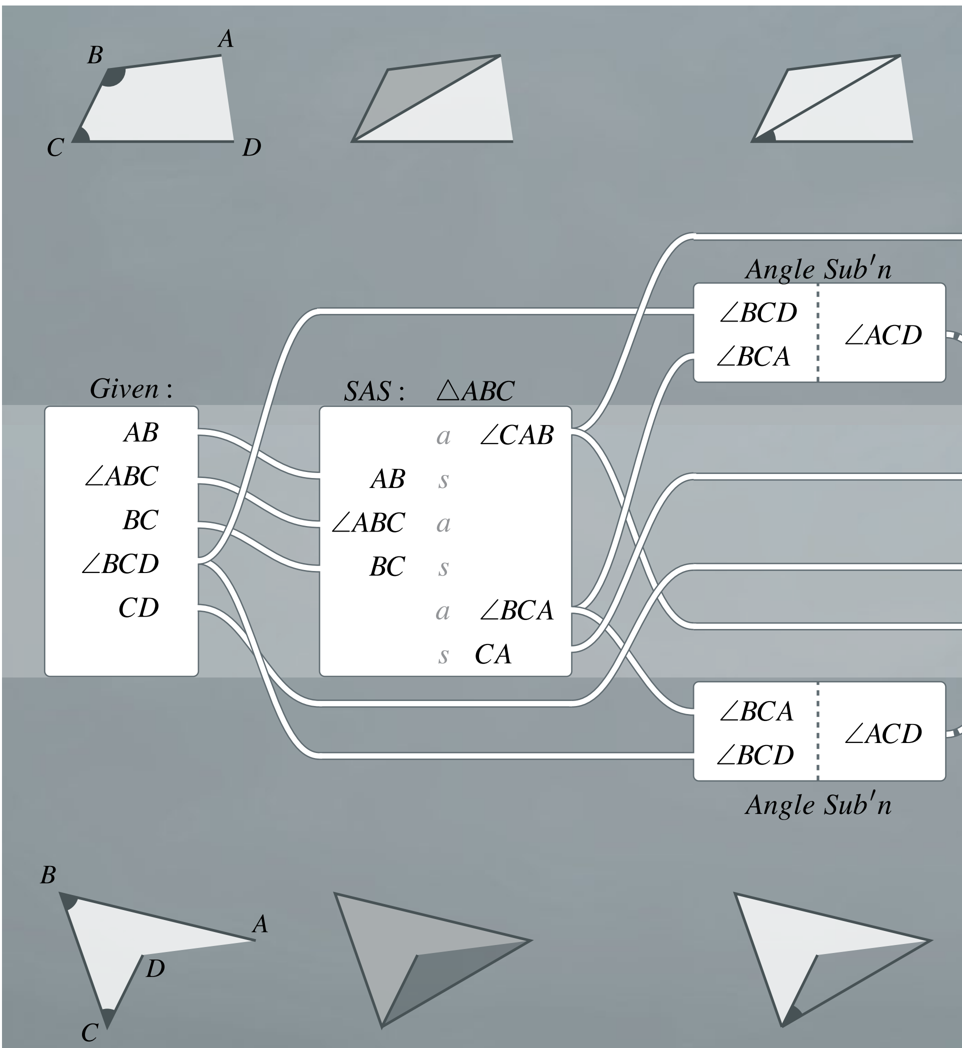
S·A·S·A·S QUADRILATERAL CONGRUENCE

If $\square ABCD$ and $\square A'B'C'D'$ are simple quadrilaterals and

$$AB \simeq A'B' \quad \angle B \simeq \angle B' \quad BC \simeq B'C' \quad \angle C \simeq \angle C' \quad CD \simeq C'D'$$

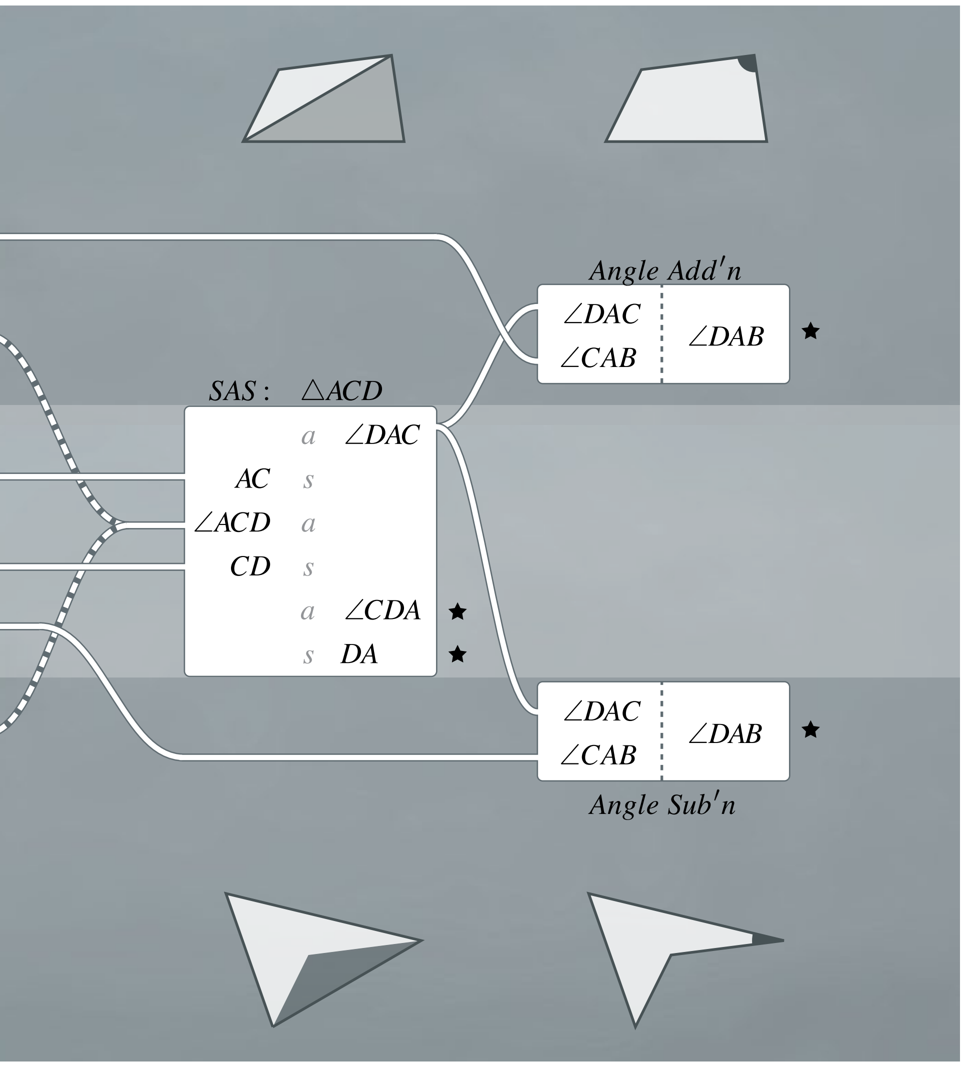
then $\square ABCD \simeq \square A'B'C'D'$.

Proof. The diagonals AC and $A'C'$ are the keys to turning this into a problem of triangle congruence. Unfortunately, we do not know whether or not those diagonals are in the interiors of their respective quadrilaterals. That means we have to tread somewhat carefully at first. Because of S·A·S, $\triangle ABC \simeq \triangle A'B'C'$. You need to pay attention to what is happening at vertex C. If AC is in the interior of the quadrilateral, then it is in the interior of $\angle BCD$ and that means $(\angle BCA) < (\angle BCD)$. Then, since $\angle B'C'A' \simeq \angle BCA$ and $\angle B'C'D' \simeq \angle BCD$, $(\angle B'C'A') < (\angle B'C'D')$. Therefore $A'C'$ must be in the interior of $\angle B'C'D'$ and in the interior of $\square A'B'C'D'$. With the same



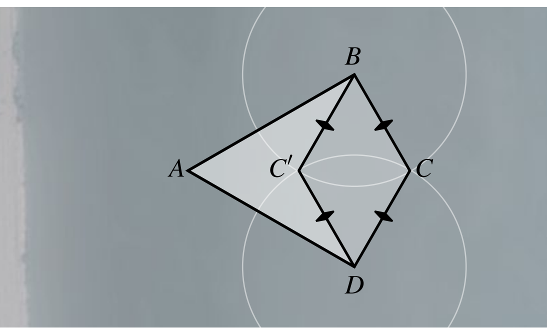
reasoning, we can argue that if AC is not in the interior of $\square ABCD$, then $A'C'$ cannot be in the interior of $\square A'B'C'D'$. So there are two cases, and the assembly of the quadrilateral from the triangles depends upon the case. My diagram of the chase through the congruences is below. I have split it, when necessary, to address the differences in the two cases. $\square$

Using essentially this same approach, you should be able to verify both the A·S·A·S·A and A·A·S·A·S quadrilateral congruences.



S·S·S·S·A

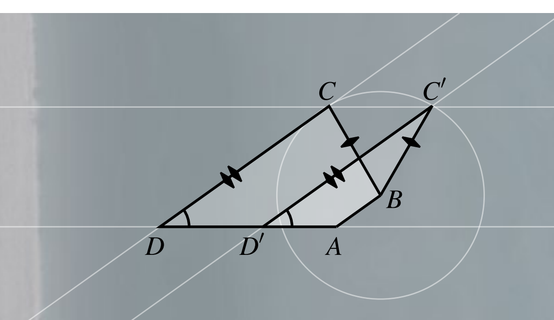
The S·S·S·S·A condition is almost enough to guarantee quadrilateral congruence. Suppose that you know the lengths of all four sides of $\square ABCD$, and you also know $\angle A$. Then $\triangle BAD$ is completely determined (S·A·S) and from that $\triangle BCD$ is completely determined (S·S·S). That still does not mean that $\square ABCD$ is completely determined, though, because there are potentially two ways to assemble $\triangle BAD$ and $\triangle BCD$ (as illustrated). One assembly creates a convex quadrilateral, the other a non-convex one. Now, there will be times when you know the quadrilaterals in question are all convex, and in those situations, S·S·S·S·A can be used to show that convex quadrilaterals are congruent.



Non-congruent quadrilaterals with matching SSSSA. One is convex; the other is not.

A·S·A·A·S, A·S·A·S·S, A·S·S·A·S, A·A·A·A·S, and S·S·S·A·A

None of these provide sufficient information to guarantee congruence and counterexamples can be found in Euclidean geometry. I will just do one of them– S·S·S·A·A, and leave the rest for you to puzzle out. In the illustration below $\square ABCD$ and $\square ABC'D'$ have correponding S·S·S·A·A but are not congruent.



Non-congruent quadrilaterals with matching SSSAA.

A·A·A·S·S

This is the intriguing one. The idea of splitting the quadrilateral into triangles along the diagonal just doesn't work. You fail to get enough information about either triangle. Yet, (as we will see) in Euclidean geometry, the angle sum of a quadrilateral has to be $360°$. Since three of the angles are given, that means that in the Euclidean realm the fourth angle is determined as well. In that case, this set of congruences is essentially equivalent to the A·S·A·S·A (which is a valid congruence theorem). The problem is that in neutral geometry the angle sum of a quadrilateral does not have to be $360°$. Because of the Saccheri-Legendre Theorem, the angle sum of a quadrilateral cannot be more than $360°$, but that is all we can say. It turns out that this *is* a valid congruence theorem in neutral geometry. The proof is a little difficult though. The argument that I want to use requires us to "drop a perpendicular". I have described this process in some of the previous exercises, but let me reiterate here.

> LEM 1
> For any line ℓ and point P not on ℓ, there is a unique line through P which is perpendicular to ℓ.

The intersection of ℓ and the perpendicular line is often called the *foot* of the perpendicular. The process of finding this foot is called *dropping a perpendicular*. I have already proven the existence part of this– the phrasing was a little different then, but my proof of the existence of right angles (in the lesson on angle comparison) constructs this perpendicular line. As for uniqueness part, I will leave that to you.

> LEM 2
> Let ℓ be a line, P a point not on ℓ, and Q the foot of the perpendicular to ℓ through P. Then P is closer to Q than it is to any other point on ℓ.

Again, I am going to pass off the proof to you. I would suggest, though, that you think about the Scalene Triangle Theorem. Now on to the main theorem.

A·A·A·S·S QUADRILATERAL CONGRUENCE
If $\square ABCD$ and $\square A'B'C'D'$ are simple quadrilaterals, and

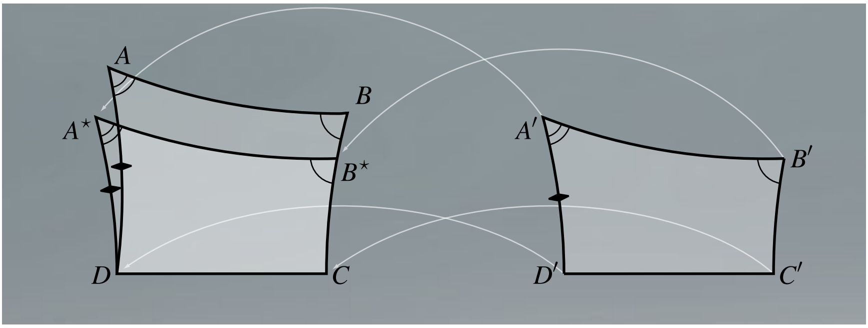
$$\angle A \simeq \angle A' \quad \angle B \simeq \angle B' \quad \angle C \simeq \angle C' \quad CD \simeq C'D' \quad DA \simeq D'A'$$

then $\square ABCD \simeq \square A'B'C'D'$.

Proof. I will use a proof by contradiction of this somewhat tricky theorem. Suppose that $\square ABCD$ and $\square A'B'C'D'$ have the corresponding congruent pieces as described in the statement of the theorem, but suppose that $\square ABCD$ and $\square A'B'C'D'$ are not themselves congruent.

Part One, in which we establish parallel lines.
I want to construct a new quadrilateral: $\square A^\star B^\star CD$ will overlap $\square ABCD$ as much as possible, but will be congruent to $\square A'B'C'D'$. Here is the construction. Locate $B^\star$ on $CB\rightarrow$ so that $CB^\star \simeq C'B'$. Note that BC and $B'C'$ cannot be congruent– if they were the two quadrilaterals would be congruent by A·A·S·A·S. As a result, in the construction, $B \neq B^\star$. The other point to place is $A^\star$. It needs to be positioned so that:

1. it is on the same side of $\leftarrow BC\rightarrow$ as A,
2. $\angle AB^\star C^\star \simeq \angle A'B'C'$, and
3. $A^\star B^\star \simeq A'B'$.



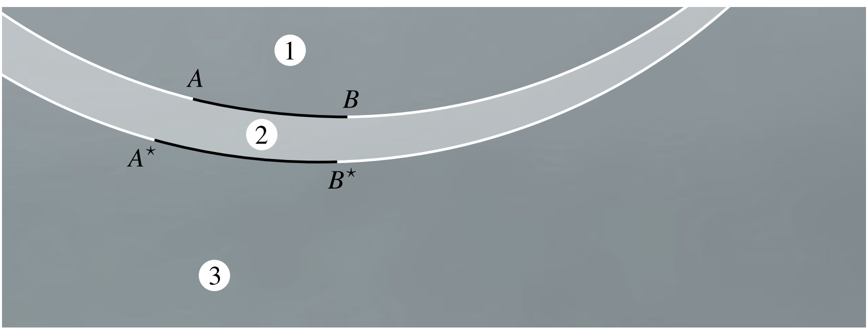
The setup for the proof of AAASS for convex quadrilaterals.

The angle and segment construction axioms guarantee that there is one and only one point that satisfies these conditions. That finishes the copying– by S·A·S·A·S, $\square A^\star B^\star CD$ and $\square A'B'C'D'$ are congruent. There is one important thing to note about this construction. Since

$$\angle A^\star B^\star C \simeq \angle A'B'C' \simeq \angle ABC,$$

the Alternate Interior Angle Theorem guarantees that $\leftarrow A^\star B^\star \rightarrow$ and $\leftarrow AB \rightarrow$ will be parallel.

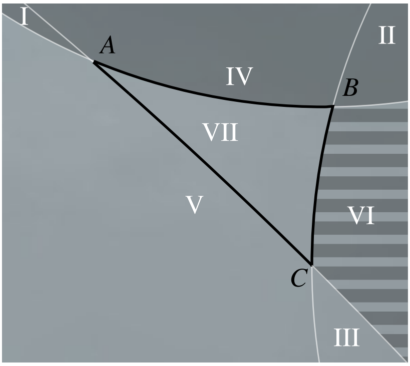
Part two, in which we determine the position of D relative to those lines. The two parallel lines $\leftarrow AB \rightarrow$ and $\leftarrow A^\star B^\star \rightarrow$ carve the plane into three regions as shown in the illustration below. The reason I mention this is



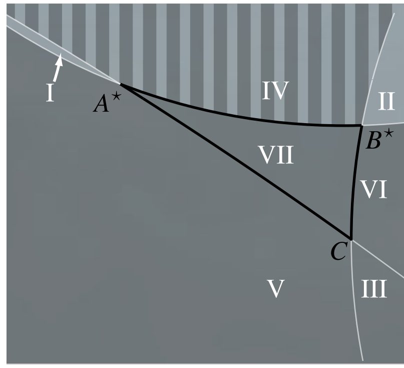
Regions between parallel lines.

that my proof will not work if D is in region 2, the region between the two parallel lines. Now it is pretty easy to show that D will not fall in region 2 if we know the two quadrilaterals are convex. If we don't know that, though, the situation gets a little more delicate, and we will have to look for possible reflex angles in the two quadrilaterals. The key thing to keep in mind is that the angle sum of a simple quadrilateral is at most 360° (a consequence of the Saccheri-Legendre Theorem), and the measure of a reflex angle is more than 180°– therefore, a simple quadrilateral will support at most one reflex angle.

Suppose that D did lie in region 2. Note that, based upon our construction, either $C * B * B^\star$ or $C * B^\star * B$, and so that means that C is *not* in region 2. Therefore, one of the two lines (either $\leftarrow AB \rightarrow$ or $\leftarrow A^\star B^\star \rightarrow$) comes between C and D while the other does not. The two cases are equivalent, so in the interest of keeping the notation reasonable, let's assume for the rest of this proof that $\leftarrow A^\star B^\star \rightarrow$ separates C and D, but that $\leftarrow AB \rightarrow$ does not. What are the implications of this? Let me refer you back to Table 2 which characterizes the possible positions of a fourth vertex of a quadrilateral in relation to the previous three.



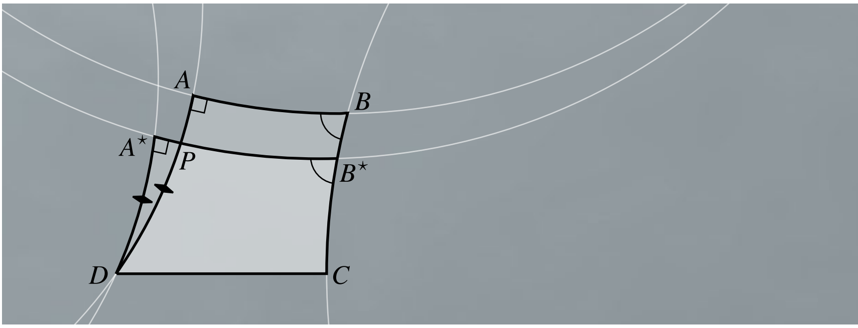
Since C and D are on the same side of $\leftarrow AB \rightarrow$, D has to be in region III, IV, or V with respect to $\triangle ABC$ (note that if D is in region VI, then $\square ABCD$ is not simple). If D is in region III, then $\square ABCD$ has a reflex angle at C. If D is in region V, then $\square ABCD$ is convex and does not have a reflex angle. And if D is in region VII, then $\square ABCD$ has a reflex angle at D.



Since C and D are on opposite sides $\leftarrow A^\star B^\star \rightarrow$, D has to be in region I or II (if D is in region IV, then $\square A^\star B^\star CD$ is not simple. If D is in region I, then $\square A^\star B^\star CD$ has a reflex angle at $A^\star$. If D is in region II, then $\square A^\star B^\star CD$ has a reflex angle at $B^\star$.

A quadrilateral can only have one reflex angle, so in $\square ABCD$ neither $\angle A$ nor $\angle B$ is reflex. In $\square A^\star B^\star CD$ one of $\angle A^\star$ or $\angle B^\star$ is reflex. Remember though that $\angle A^\star \simeq \angle A$ and $\angle B^\star \simeq \angle B$. This is a contradiction– obviously two angles cannot be congruent if one has a measure over 180° while the other has a measure less than that. So now we know that D cannot lie between $\leftarrow AB \rightarrow$ and $\leftarrow A^\star B^\star \rightarrow$ and so *all* the points of $\leftarrow AB \rightarrow$ are on the opposite side of $\leftarrow A^\star B^\star \rightarrow$ from D.

Part Three, in which we measure the distance from D to those lines.
I would like to divide the rest of the proof into two cases. The first case deals with the situation when $\angle A$ and $\angle A^\star$ (which are congruent) are right angles. The second deals with the situation where they are not.

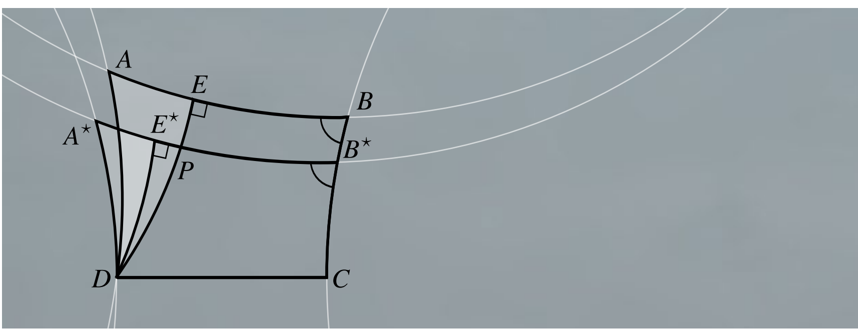


Case 1: the angle at A is a right angle.

Case 1. $(\angle A) = (\angle A^\star) = 90^\circ$.
Since D and A are on opposite sides of $\leftarrow A^\star B^\star \rightarrow$, there is a point P between A and D which is on $\leftarrow A^\star B^\star \rightarrow$. Then

$$|DP| < |DA| = |DA^\star|.$$

But that can't happen, since $A^\star$ is the closest point on $\leftarrow A^\star B^\star \rightarrow$ to D.



Case 2: the angle at A is not a right angle.

Case 2. $(\angle A) = (\angle A^\star) \neq 90^\circ$.
The approach here is quite similar to the one in Case 1. The difference is that we are going to have to make the right angles first. Locate E and $E^\star$, the feet of the perpendiculars from D to $\leftarrow AB \rightarrow$ and $\leftarrow A^\star B^\star \rightarrow$, respectively. Please turn your attention to triangles $\triangle DAE$ and $\triangle DA^\star E^\star$. In them,

$$AD \simeq A^\star D \quad \angle A \simeq \angle A^\star \quad \angle E \simeq \angle E^\star.$$

By A·A·S, they are congruent, and that means that $DE \simeq DE^\star$. But that creates essentially the same problem that we saw in the first case. Since D

and E are on opposite sides of $\leftarrow A^\star B^\star \rightarrow$, there is a point P between D and E which is on $\leftarrow A^\star E^\star \rightarrow$. Then

$$|DP| < |DE| = |DE^\star|.$$

Again, this cannot happen, as $E^\star$ should be the closest point to D on $\leftarrow A^\star E^\star \rightarrow$.

In either case, we have reached a contradiction. The initial assumption, that $\square ABCD$ and $\square A'B'C'D'$ are *not* congruent, must be false.

$\square$

Exercises

1. A convex quadrilateral with two pairs of congruent adjacent sides is called a *kite*. Prove that the diagonals of a kite are perpendicular to one another.

2. Prove the A·S·A·S·A, and A·A·S·A·S quadrilateral congruence theorems.

3. Prove the S·S·S·S·A quadrilateral congruence theorem for *convex* quadrilaterals.

4. Provide Euclidean counterexamples for each of A·S·A·A·S, A·S·A·S·S, A·S·S·A·S, and A·A·A·A·S.

5. Here is another way that you could count words: there are four angles and four sides, a total of eight pieces of information, and you need to choose five of them. That means there are

$$\binom{8}{5} = \frac{8!}{5!(8-5)!} = 56$$

possibilities. That's quite a few more than the $2^5 = 32$ possibilities that I discussed. Resolve this discrepancy and make sure that I haven't missed any congruence theorems.