Methods of Integration

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Example: $\int x^3 e^x\,dx$

$$\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx \quad \boxed{1}$$

$$= x^3 e^x - 3\left[x^2 e^x - 2\int x e^x\,dx\right] \quad \boxed{2}$$

$$= x^3 e^x - 3\left[x^2 e^x - 2\left[x e^x - \int e^x\,dx\right]\right] \quad \boxed{3}$$

$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C$$

1

$u = x^3 \qquad dv = e^x\,dx$

$du = 3x^2\,dx \qquad v = e^x$

2

$u = x^2 \qquad dv = e^x\,dx$

$du = 2x\,dx \qquad v = e^x$

3

$u = x \qquad dv = e^x\,dx$

$du = dx \qquad v = e^x$

Integration by parts is useful when the integrand is a product of two different kinds of pieces. For instance, an exponential term times a trigonometric term, or a logarithmic term times an algebraic term.

Note that it may be necessary to do the procedure more than once.

If after a few iterations you end up back at the starting integral, you may be able to solve the integral by gathering the occurrences of that integral on one side of the equation.

Rule of Thumb for choosing u (expressions at the top of the list tend to make better choices of u):

Inverse trigonometric
Logarithmic
Algebraic
Trigonometric
Exponential expression

Trigonometric Integrals

$$\int \sin^m x \cos^n x \, dx$$

If n is odd, use the identity

$$\sin^2 x + \cos^2 x = 1,$$

to convert all but one cosine terms to sine. Then substitute $u = \sin x$. Similarly, if m is odd, convert the sine terms to cosine, leaving one sine term, and substitute $u = \cos x$.

If both powers are even, then you must use a combination of "double angle" identities to simplify the integrand. Begin with:

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

Then use:

$$\sin^2(x) = \frac{1}{2}[1 - \cos(2x)]$$

$$\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$$

Examples:

$$\begin{aligned}
\int \sin^5 x \cos^2 x \, dx &= \int \sin x \cdot \sin^4 x \cos^2 x \, dx \\
&= \int \sin x (1 - \cos^2 x)^2 \cos^2 x \, dx \\
&= \int \sin x (\cos^2 x - 2\cos^4 x + \cos^6 x) \, dx \\
&= \int (-u^2 + 2u^4 - u^6) \, du \\
&= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C \\
&= -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C
\end{aligned}$$

$$\begin{aligned}
u &= \cos x \\
du &= -\sin x \, dx
\end{aligned}$$

$$\begin{aligned}
\int \sin^4 x \cos^4 x \, dx &= \int \left(\frac{1}{2}\sin(2x)\right)^4 dx \\
&= \frac{1}{16}\int \sin^4(2x) \, dx \\
&= \frac{1}{16}\int \left(\frac{1}{2}[1 - \cos(2x)]\right)^2 dx \\
&= \frac{1}{64}\int \left(1 - 2\cos(2x) + \cos^2(2x)\right) dx \\
&= \frac{1}{64}\left[x - \sin(2x) + \frac{1}{2}\int (1 + \cos(4x)) \, dx\right] \\
&= \frac{1}{64}\left[x - \sin(2x) + \frac{1}{2}\left(x + \frac{\sin(4x)}{4}\right)\right] + C \\
&= \frac{1}{64}\left[\frac{3}{2}x - \sin(2x) + \frac{1}{8}\sin(4x)\right] + C
\end{aligned}$$

$$\int \tan^m x \sec^n x \, dx$$

If n is even, use the identity

$$\sec^2 x = 1 + \tan^2 x$$

to convert all but two of the secants into tangents. Then substitute $u = \tan x$. If m is odd, convert all but one of the tangents into secant, and substitute $u = \sec x$.

Similar strategies work for combinations of powers of cotangent and cosecant.

Example:

$$\int \tan^3 x \sec^2 x \, dx$$

$$\begin{aligned}
\int \tan^3 x \sec^2 x \, dx &= \int \tan x (\sec^2 x - 1) \sec^2 x \, dx \\
&= \int (\sec^3 x - \sec x) \sec x \tan x \, dx \\
&= \int (u^3 - u) \, du \\
&= \frac{u^4}{4} - \frac{u^2}{2} + C \\
&= \frac{\sec^4 x}{4} - \frac{\sec^2 x}{2} + C
\end{aligned}$$

$$u = \sec x$$
$$du = \sec x \tan x \, dx$$

multiple angles

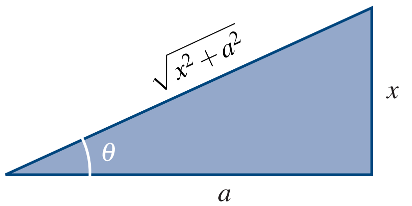
Use the identities (derived from addition formulas for sine and cosine):

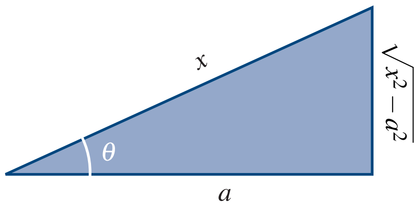
$$\int \sin(mx) \sin(nx) \, dx \qquad \sin A \sin B = \frac{1}{2} [\sin(A - B) - \cos(A + B)]$$

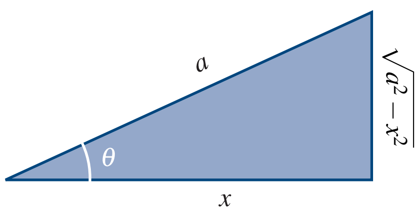
$$\int \cos(mx) \cos(nx) \, dx \qquad \cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

$$\int \sin(mx) \cos(nx) \, dx \qquad \sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)]$$

Trigonometric Substitution







Trigonometric substitution is useful when the integrand has a term of the form x^2+a^2, x^2-a^2, or a^2-x^2. This term is often (but not always) inside of a square root or in the denominator of a fraction. There are essentially three cases, all involving replacing algebraic expressions with trigonometric expressions.

Note that it may be necessary to complete the square.

Examples:

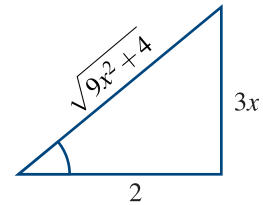
$$\int \frac{dx}{\sqrt{9x^2+4}} = \int \frac{(2/3)\sec^2\theta}{2\sec\theta}\,d\theta \quad \boxed{1}$$

$$= \frac{1}{3}\int \sec\theta\,d\theta$$

$$= \frac{1}{3}\ln|\sec\theta+\tan\theta|+C$$

$$= \frac{1}{3}\ln\left|\frac{\sqrt{9x^2+4}}{2}+\frac{3x}{2}\right|+C$$

1



$x = \frac{2}{3}\tan\theta$

$\sqrt{9x^2+4} = 2\sec\theta$

$dx = \frac{2}{3}\sec^2\theta\,d\theta$

$$\int \frac{dx}{\sqrt{x^2+2x+10}} = \int \frac{dx}{\sqrt{(x+1)^2+3^2}}$$

$$= \int \frac{du}{\sqrt{u^2+9}} \quad \boxed{2}$$

$$= \int \frac{3\sec^2\theta}{3\sec\theta}\,d\theta \quad \boxed{3}$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta+\tan\theta|+C$$

$$= \ln\left|\frac{\sqrt{u^2+9}}{3}+\frac{u}{3}\right|+C$$

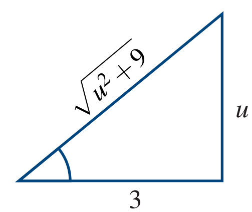
$$= \ln\left|\frac{\sqrt{(x+1)^2+9}}{3}+\frac{x+1}{3}\right|+C$$

2

$u = x+1$

$du = dx$

3



$u = 3\tan\theta$

$\sqrt{u^2+9} = 3\sec\theta$

$du = 3\sec^2\theta\,d\theta$

Partial Fractions

If the degree of the numerator is higher than the degree of the denominator, before beginning the partial fractions procedure, you must perform polynomial long division.

The first step in determining the partial fractions decomposition is to factor the denominator. While *in practice* this may be very difficult, *in theory* it is possible to factor the denominator into the following types of forms:

1. $x+a$
2. x^2+ax+b
3. $(x+a)^n$
4. $(x^2+ax+b)^n$

The corresponding terms in the partial fraction decomposition:

1. $\dfrac{A}{x+a}$
2. $\dfrac{Ax+B}{x^2+ax+b}$
3. $\dfrac{A_1}{x+a}+\dfrac{A_2}{(x+a)^2}+\cdots+\dfrac{A_n}{(x+a)^n}$
4. $\dfrac{A_1x+B_1}{x^2+ax+b}+\dfrac{A_2x+B_2}{(x^2+ax+b)^2}+\cdots+\dfrac{A_nx+B_n}{(x^2+ax+b)^n}$

Use linear algebra to find the values of the coefficients in the numerators of these fractions.

Example:

$$\int \frac{5x^2+3}{x^3+x}\,dx$$

$$\begin{aligned}\frac{5x^2+3}{x^3+x} &= \frac{A}{x}+\frac{Bx+C}{x^2+1}\\ &= \frac{A(x^2+1)+x(Bx+C)}{x^3+x}\end{aligned}$$

$$5x^2+3 = Ax^2+A+Bx^2+Cx$$

$$\Longrightarrow \begin{cases} 5 = A+B \\ 3 = A \implies B = 2 \\ 0 = C \end{cases}$$

$$\begin{aligned}\int \frac{5x^2+3}{x^3+x}\,dx &= \int \left(\frac{3}{x}+\frac{2x}{x^2+1}\right)dx\\ &= 3\ln|x| + \int \frac{du}{u}\\ &= 3\ln|x| + \ln|u| + C\\ &= 3\ln|x| + \ln(x^2+1) + C\end{aligned}$$

$$\boxed{\begin{aligned}u &= x^2+1\\ du &= 2xdx\end{aligned}}$$