Lesson: Errors and Norms

Errors, Norms and Vector Porperties

In many numerical problems, it is necessary to know how close a vector (or matrix) is to zero. This is important so that the error of a numerical method can be analyzed in some way. In order to do so, we introduce the norm of a vector and also the norm of a matrix (sometimes the two are related).

1. Types of Errors

The absolute error is the norm of the difference or

$$e_{abs} = \left\| y^* - y_n \right\|$$

where y^* is the exact answer and y_n is the approximation.

The relative error is calculated by

$$e_{rel} = \frac{\left\| y^* - y_n \right\|}{\left\| y^* \right\|}$$

In both of these equation, any of the norms that we discuss later may be used.

Example: Consider the following algorithm for approximating the value of $\sqrt{2}$.

$$y_n = \frac{y_{n-1}}{2} + \frac{1}{y_{n-1}}$$

Start with y_0 as some initial guess of $\sqrt{2}$. If $y_0 = 1$, then $y_1 = \frac{1}{2} + \frac{1}{1} = 1.5$ and so forth. Hopefully $\lim_{n\to\infty} y_n = y^* = \sqrt{2}$.

The following matlab m-file runs two iterations of this algorithm so that we can see how it does by looking at the errors:

```
function y = sqrttwo(y0)
% sqrttwo.m -- approximating sqrt(2)
% y0 is the initial guess fo the sqrt(2)
% This code will only run 2 iterations

y = y0;
for i = 1:2
 = y/2 + 1/y;
end
```

Running it with $y_0 = 3$ we get the following

```
>> y = sqrttwo(3)

y =
   1.4621

>> ystar = sqrt(2);
>> err_abs = abs(y - ystar)

err_abs =
   0.0479

>> err_rel = abs(y - ystar) / abs(ystar)

err_rel =

   0.0339
```

2. Vector Norms

   $\|\bullet\|$ is a norm if

   1. For **v** a vector, $\|v\| \geq 0$ and $\|v\| = 0$ when **v** = **0**.
   2. $\|kv\| = |k|\|v\|$ where k is a real scalar.
   3. For two vectors **v** and **w,** $\|v + w\| \leq \|v\| + \|w\|$ (triangle inequality)

   Here are some common norms:

   $$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \text{(the } l_1 \text{ norm)}$$

   $$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \qquad \text{(the } l_2 \text{ norm)}$$

   $$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \qquad \text{(the } l_\infty \text{ norm)}$$

   The reason that norms are important in numerical methods is because we want to see how close our errors are to zero.

   For example, if x^* is the exact solution of some mathematical problem and x_n is an approximation of x^* then we could define the error in this solution to be $e_n = x^* - x_n$. Then the question is: <u>How close is this error to zero</u>?

   The **norm** command can be used to calculate various norms of vectors:

```
>> x = rand(1,4)
```

```
x =
   0.9501    0.2311    0.6068    0.4860
```

>> norm(x,1)   % the l_1 norm: equivalent to sum(abs(x))

```
ans =
   2.2741
```

>> norm(x,2)   % the l_2 norm: equivalent to sum(x.^2).^(1/2)

```
ans =
   1.2492
```

>> norm(x,inf)   % the l_∞ norm: equivalent to max(abs(x))

```
ans =
   0.9501
```

For more information about the **norm** command type **help norm**. Typically, we will use the l_2 norm or **norm**(x,2).

3. Matrix Norms
   There are generally two types of norms for matrices. The first type is one which satisfies all the definitions of a typical norm. The second type is called a subordinate norm which is closely related to a particular vector norm.

   The matrix norm subordinate to the vector norm $\|\bullet\|$ is defined in the following way:

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

   This subordinate norm has all the properties of a regular norm with the following added property:

$$\|AB\| \le \|A\|\|B\|$$

   Because of the way that they are defined, matrix norms are not typically very easy to calculate. Two that are easy to calculate are the l_1 and l_∞ subordinate norms:

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{i,j}| \qquad \text{maximum column sum}$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{i,j}| \qquad \text{maximum row sum}$$

Again these norms can be calculated with the **norm** command.

```
>> A = [-1 2; 3 4]

A =
   -1    2
    3    4

>> norm(A,1)

ans =
   6

>> norm(A,inf)

ans =

   7
```

The l_2 subordinate norm can also be calculate by Matlab. It's definition is not "quite as clear" however. It is often defined to be equal to the square root of the largest eigenvalue of $A^T A$.

```
>> norm(A,2)            % the equivalent of sqrt(max(eig(A'*A)))

ans =
   5.1167
```