SENSITIVITY EQUATIONS FOR THE DESIGN OF CONTROL SYSTEMS

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ABSTRACT
Systematic strategies for optimal actuator and sensor locations require £nding extrema of control performance measures. When the control is designed for a distributed parameter system, these performance measures frequently involve the kernel of the Riccati operator or that of the feedback operator. For example, measuring the optimal linear quadratic regulator (LQR) cost over a range of initial data involves the Riccati operator. To aid in the design process, we consider sensitivity equations for Riccati and Chandrasekhar equations. The latter is well-suited for computing feedback kernels when there are a small number of control inputs and control outputs. As we demonstrate, the sensitivity of these kernels to actuator positions can lead to ef£cient computation of gradients for optimization algorithms. Numerical examples corresponding to placing an actuator in the heat equation are provided.

KEY WORDS
Optimization, Actuator Placement, Sensitivity Equations, Distributed Parameter Systems

1 Introduction

Since the genesis of feedback control, design engineers have wrestled with the issue of actuator and sensor placement. A well positioned actuator can either provide more control authority with a £xed amount of control energy, or provide the necessary control authority with minimum energy. Recent surveys of the optimal actuator design problem can be found in [6] and [12]. Choosing actuator positions to minimize the optimal LQR cost (cf. [3]) have been considered recently in distributed parameter control problems [2, 4]. Aside from being a natural design objective to consider (minimizing it is the premise for LQR control), it can be expressed in a convenient mathematical form and is thus amenable to optimization.

In this paper, we consider a strategy for ef£cient optimization of control systems using sensitivity equations for fast gradient calculations. Our primary motivating problem is positioning actuators to minimize the norm of the Riccati operator. However, our discussion holds for a number of related control measures (e.g. optimal min-max Riccati norm [4]). We begin by setting up an actuator placement problem involving the heat equation. The problem is to determine the best location of a heat sink (source) in the system. Two popular approaches for solving the distributed parameter control problem are considered, one based on Riccati equations [9, 1, 7] and the other based on Chandrasekhar equations [11, 10]. The computational complexity of these control problems is low enough where plotting the entire design objective function over a parameter range is feasible.

This provides a means of testing our approach which is based on computing gradients using sensitivity equations. The high cost of solving a distributed parameter control problem is practice, necessitates the development of ef£cient gradient based optimization algorithms and fast gradient calculations. Sensitivity equations are extremely useful in this respect, since they typically provide derivative information (and more) in a fraction of the cost of computing a feedback control law. These are practical when a small number of design parameters are used (as in our actuator placement problems). We develop and discuss sensitivity equations for both Riccati and Chandrasekhar equations as well as provide the associated abstract PDEs (useful for numerical procedures such as adaptive mesh re£nement). Finally numerical results demonstrating the accuracy and effectiveness of this optimization approach are included.

2 Problem Statement

To illustrate our ideas for optimal design of control systems and ef£cient computational tools, we introduce a control problem using the one-dimensional heat equation. Consider

$$\begin{aligned} z_t(t,x) &= \kappa\, z_{xx}(t,x) + b(x;\gamma)u(t) \\ z(t,0) &= 0, \quad z(t,1) = 0, \\ z(0,x) &= z_0(x) \end{aligned} \tag{1}$$

for $t > 0$ and $x \in (0,1)$, where k is the thermal diffusivity of the material. We describe the control source by

$$b(x;\gamma) = e^{-(x-\gamma)^2} \quad \text{for} \quad x \in [0,1], \gamma \in (.1,.9). \tag{2}$$

Thus the parameter γ is used to locate the control actuator. Our design objective is to £nd the value of γ that optimizes some control measure.

To take advantage of distributed parameter control theory, we formulate this boundary value problem (1) as an abstract state space model. As is common in this £eld, we de£ne $z(\cdot)$ to be an element of a Hilbert space and recast (1) as

$$\begin{gathered} \dot{z}(t) = Az(t) + Bu(t) \\ z(0) = z_0 \end{gathered} \tag{3}$$

where A is the differential operator

$$[Az](t) = \kappa \frac{d^2 z}{dt^2}(t) \tag{4}$$

de£ned on the domain $\mathcal{D}(A) \equiv H_0^1(0,1) \cap H^2(0,1)$ and B is de£ned by

$$[Bu](t) = e^{-(x-\gamma)^2} u(t). \tag{5}$$

3 Distributed Parameter LQR Problem

We consider a LQR control problem de£ned by minimizing the cost function

$$J(u(\cdot)) = \int_0^\infty \{\langle y(t), y(t)\rangle + \langle u(t), u(t)\rangle\}\, dt \tag{6}$$

subject to the constraint (3), where $y(t) = Cz(t)$ is a controlled output function. To make this discussion more interesting, we consider a few control output functions. We minimize the control output over a given interval $[a, b],\ a \in (0, 1),\ b \in (0, 1),\ a < b$:

$$y(t) = \int_a^b z(t, x)\, dx. \tag{7}$$

3.1 Riccati Equation

Under suitable conditions [9], the LQR problem has the optimal control given by feedback of the form

$$u_{opt}(t) = -Kz(t) = -B^*\Pi z(t) \tag{8}$$

where K is the feedback gain operator. Here Π is the (weak) solution to the algebraic Riccati Equation (ARE)

$$A^*\Pi + \Pi A - \Pi BB^*\Pi + Q = 0 \tag{9}$$

and $Q = C^*C$.

It has been shown [9] that the feedback operator K has the form

$$K\phi = \int_0^1 h(x)\phi(x)\, dx \tag{10}$$

where the kernel $h(\cdot)$ belongs to $L^2(0, 1)$. $h(\cdot)$ is called the functional gain.

3.2 Chandrasekhar Equations

Chandrasekhar methods may be used to bypass the step of solving the Riccati equations [5, 8]. The Chandrasekhar equations are formally de£ned by

$$-\dot{K}(t) = B^*L^*(t)L(t) \tag{11}$$

$$-\dot{L}(t) = L(t)[A - BK(t)] \tag{12}$$

with £nal conditions

$$K(T) = 0 \qquad \text{and} \qquad L(T) = C. \tag{13}$$

Under suitable conditions, it has been shown that $K(t) \to K$ as $t \to -\infty$. Similarly, Sorine [11] has shown

$$K(t)\phi = \int_0^1 h(t, \xi)\phi(\xi)\, d\xi \tag{14}$$

and

$$L(t)\phi = \int_0^1 l(t, \xi)\phi(\xi)\, d\xi. \tag{15}$$

4 Computing Functional Gains

There are two approaches to £nd feedback control laws for (linear) distributed parameter systems: Introduce *approximations* to the system, then use £nite dimensional *control* techniques or develop representations for the *control* laws as the solutions to a partial differential equation (PDE), then *approximate* the system. These techniques are referred to as "approximate-then-control" and "control-then-approximate," respectively. The discrete and PDE versions of the Chandrasekhar and Riccati equations may be computationally equivalent, but are conceptually different.

4.1 Riccati Equation

We can approximate (3) by projecting it onto a suitable £nite element subspace, cf. [7]. Then we get the approximating system

$$\dot{z}^N(t) = A^N z^N(t) + B^N u(t) \tag{16}$$

which should be interpreted in the weak sense. Thus using £nite element formulations we get a discrete equation which can be expressed as

$$\mathbf{M}\dot{\mathbf{z}}(t) = -\mathbf{S}\mathbf{z}(t) + \mathbf{E}u(t) \tag{17}$$

where the matrices **M** and **S** are sparse and **E** has low rank. Placing the above system in canonical form yields

$$\begin{aligned} \dot{\mathbf{z}}(t) &= -\mathbf{M}^{-1}\mathbf{S}\mathbf{z}(t) + \mathbf{M}^{-1}\mathbf{E}u(t) \\ &\equiv \mathbf{A}\mathbf{z}(t) + \mathbf{B}u(t) \end{aligned} \tag{18}$$

and $\mathbf{z}(0) = \mathbf{z}_0$. Then we solve the Riccati equation

$$\mathbf{\Pi A} + \mathbf{A^T \Pi} - \mathbf{\Pi B B^* \Pi} + \mathbf{Q} = 0 \tag{19}$$

for $\mathbf{\Pi}$, a symmetric positive de£nite matrix. This can be done using ef£cient software such as MATLAB'S `lqr`. Thus the approximate feedback operator will be of the form

$$\mathbf{K} = \mathbf{B^T \Pi} \qquad (i.e. \quad \mathbf{u}(t) = -\mathbf{Kz}(t)) \tag{20}$$

and the functional gain of the form

$$\mathbf{H} = \mathbf{M^{-1} K^T}. \tag{21}$$

There is a continuous PDE version of the Riccati equation, R-PDE, but due to space limitation and lack of ef£ciency in using the R-PDE to compute functional gains, we omit discussing it here.

4.2 Chandrasekhar Equations

For problems in which the number of controls *m* and observations *p* are dominated by the dimension of the approximating state, N, it is more ef£cient to use the Chandrasekhar factorization to simplify the calculations. This is due to the fact that there are $\frac{N(N-1)}{2}$ unknowns in $\mathbf{\Pi}$ and only $N \times m$ unknowns in $\mathbf{K}$.

The feedback matrix $\mathbf{K}$ may be found by solving

$$\begin{aligned} -\dot{\mathbf{K}}(t) &= \mathbf{B}^* \mathbf{L}^*(t) \mathbf{L}(t) \\ -\dot{\mathbf{L}}(t) &= \mathbf{L}(t)[\mathbf{A} - \mathbf{BK}(t)] \end{aligned} \tag{22}$$

with initial conditions $\mathbf{K}(T) = 0$ and $\mathbf{L}(T) = \mathbf{C}$. This method does not adapt itself to mesh re£nement.

Instead we develop partial differential equations for the Chandrasekhar equations by essentially combining (11), (12), (13), (14), and (15).

Theorem 1 *The kernels $h(\cdot,\cdot)$ and $l(\cdot,\cdot)$ from (14) and (15) satisfy the following Chandrasekhar partial differential equations (C-PDE):*

$$-\frac{\partial}{\partial t} h(t,\xi) = l(t,\xi) \int_0^1 e^{-(x-\gamma)^2} l(t,x)\, dx \tag{23}$$

$$\begin{aligned} -\frac{\partial}{\partial t} l(t,\xi) = \kappa \frac{\partial^2}{\partial \xi^2} l(t,\xi) \\ - h(t,\xi) \int_0^1 e^{-(x-\gamma)^2} l(t,x)\, dx \end{aligned} \tag{24}$$

with £nal conditions

$$h(T,\xi) = 0 \quad \text{and} \quad l(T,\xi) = c(\xi). \tag{25}$$

One could combine (22) with (23) and (24) to develop adaptive mesh re£nement strategies.

5 Actuator Placement Problem

The dependence of B upon γ provides a design variable for optimal actuator location. We now consider possible design objectives for the optimal actuator location problem.

Since the area where the heat sink is concentrated is parameter dependent, one possible design objective is to [2]: £nd γ that minimizes

$$\max_{z_0 \in \bar{Z}} J(u_{opt}, z_0) \tag{26}$$

for initial data in a certain set of functions $\bar{Z}$ and γ in an acceptable range. For the LQR problem, this is equivalent to:

$$\min_{\gamma \in (.1,.9)} \max_{z_0 \in \bar{Z}} \langle z_0, \mathbf{\Pi}(\gamma) z_0 \rangle. \tag{27}$$

For the case where $\| \; z_0 \; \| = 1$, we have the problem of minimizing the norm of $\mathbf{\Pi}(\gamma)$ over all possible $\gamma \in (.1, .9)$.

It has been conjectured [2] that if $b > 0$, we have the equivalent problem of maximizing

$$\int_0^1 [h(x;\gamma)]^2 \, dx \tag{28}$$

over all admissible $\gamma's$, where $h(\cdot)$ is the kernel of K obtained from the Chandrasekhar equations.

6 Sensitivity Equations

The concern is that it is not very ef£cient to optimize $\| \; \mathbf{\Pi} \; \|$ or $\| \; h \; \|$ by plotting the entire objective function. Thus we need an ef£cient gradient-based algorithm to optimize the objective function. We use sensitivity equations to compute the gradients needed to optimize the objective function.

6.1 Riccati Equation

We formally differentiate the parameter dependent Riccati equation (19) with respect to the design variable γ. After rearranging we get a Lyapunov matrix equation:

$$\mathbf{DX} + \mathbf{XD^T} = -\mathbf{C} \tag{29}$$

where

$$\mathbf{D} = (\mathbf{A} - \mathbf{BB^T \Pi})^\mathbf{T} \tag{30}$$

$$\mathbf{C} = -\mathbf{\Pi}(\frac{\partial \mathbf{B}}{\partial \gamma} \mathbf{B^T} - \mathbf{B} \frac{\partial \mathbf{B^T}}{\partial \gamma}) \mathbf{\Pi} \tag{31}$$

$$\mathbf{X} = \frac{\partial \mathbf{\Pi}}{\partial \gamma}. \tag{32}$$

We can solve for $\frac{\partial \mathbf{\Pi}}{\partial \gamma}$ using MATLAB'S `lyap`. Then

$$\frac{\partial \mathbf{H}}{\partial \gamma} = \mathbf{M^{-1}}(\frac{\partial \mathbf{B^T}}{\partial \gamma} \mathbf{\Pi} + \mathbf{B^T} \frac{\partial \mathbf{\Pi}}{\partial \gamma}). \tag{33}$$

6.2 Chandrasekhar Equations

We derive the sensitivity equations for γ by formally differentiating the weak form of the C-PDE (23), (24), and (25) with respect to the parameter γ. We do not give the equations due to space limitations.

Thus we can optimize our objective functions

$$\min_{\gamma \in (.1,.9)} \parallel \mathbf{\Pi}(\gamma) \parallel, \tag{34}$$

or

$$\max_{\gamma \in (.1,.9)} \int_0^1 [h(x;\gamma)]^2 \, dx \tag{35}$$

without plotting the entire objective functions. Another important observation about the Chandrasekhar PDE approach is that it allows for parallel tools and adaptive mesh re£nement.

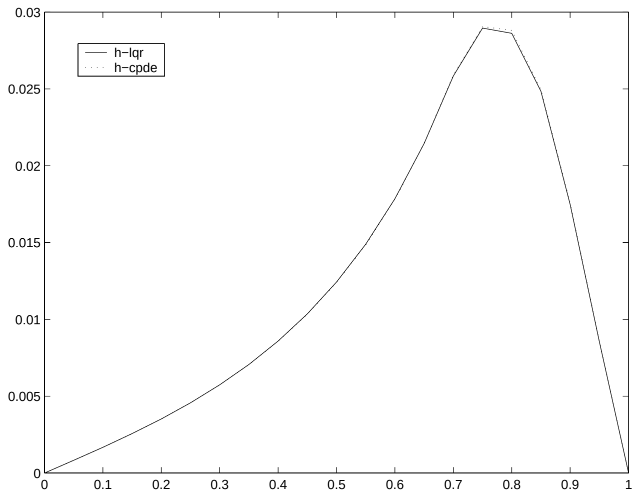

Figure 1. Functional gains

7 Numerical Results

We use the Galerkin procedure with continuous linear elements to approximate the PDE's above. Then the right hand sides are evaluated by the Crank Nicolson £nite difference scheme and the equations are solved by Newton's Method.

We test our solutions using the initial condition $z(0, x) \equiv 1$. The £nal time $T = 6$ and time step, $\Delta t = .05$ are used in our calculations. Relatively coarse meshes of 10 and 20 elements are used to obtain our approximations.

The functional gain, $h(\cdot)$, via MATLAB'S `lqr` and via C-PDE are plotted in Figure 1 when we are minimizing the average temperature over the interval [.7, .9] and the actuator location is stationary at $\gamma = 0.5$. There is really no noticable difference in the plots. The C-PDE method requires a considerable amount of time more than MATLAB'S `lqr`. We feel that this differential would lessen if the number of controls and observations were increased.

The sensitivities, $\frac{\partial \mathbf{H}}{\partial \gamma}$, and $\frac{\partial h}{\partial \gamma}$ are plotted in Figures 2 and 3 respectively when we are minimizing the average temperature over the interval [0, 1] and the actuator location is stationary at $\gamma = 0.5$. Notice that both methods (Riccati and Chandrasekhar) yield similar sensitivity results.

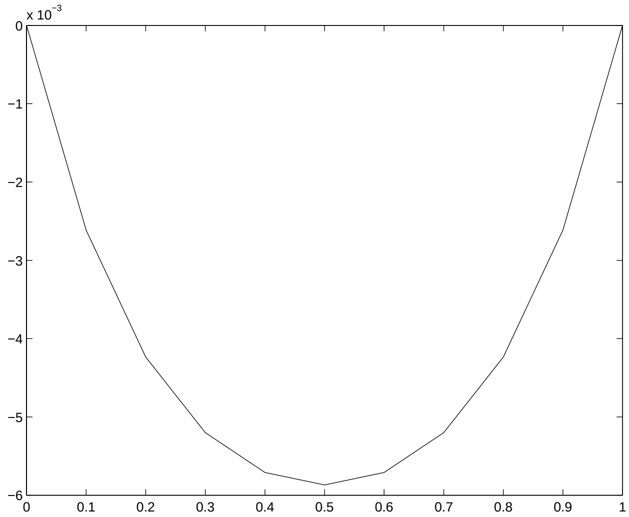

Figure 2. Sensitivity via Riccati

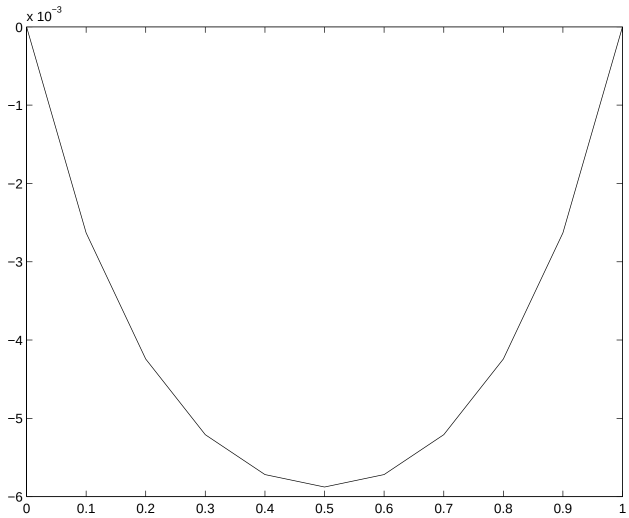

Figure 3. Sensitivity via C-PDE

The CPU time to get Figure 2 is .5010 seconds, since we have to solve one Riccati and one Lyapunov Equation. The Lyapunov equation solve is essentially free. Plots of $\frac{\partial H}{\partial \gamma}$, over the interval [0, 1] via £nite differences are in Figures 4 and 5. The CPU time for these two plots is ..9120 seconds. This is due to the fact that we had to solve 2 Riccati Equations. In Figure 4 we used $\Delta\gamma = .01$ and in Figure 5 we used $\Delta\gamma = .001$. So not only does it take longer using £nite differences, but you must also worry about the step size of γ. Thus, using sensitivity equations to compute $\frac{\partial \mathbf{H}}{\partial \gamma}$ is more precise than £nite differences.

In Figure 6, we plot the norm of the Riccati matrix as we vary the parameter γ from 0.1 to 0.9. Here we are minimizing the average temperature over the interval [.7, .9].

The objective looks like a nice quadratic function. Figure 7 plots $\frac{\partial \|\mathbf{\Pi}\|}{\partial \gamma}$ as we vary γ. Notice that the mini-

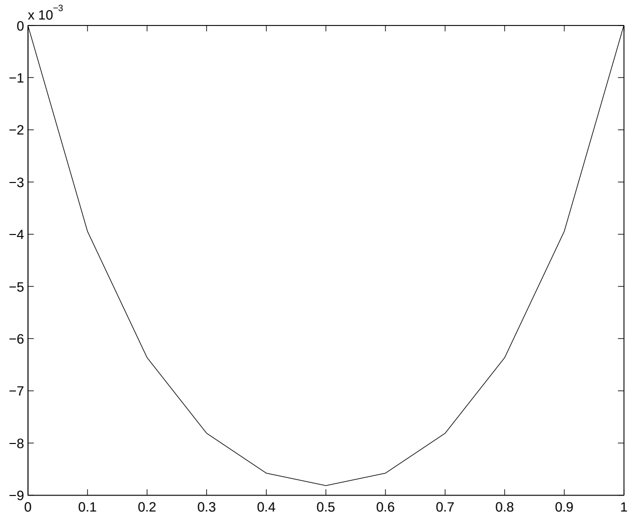

Figure 4. Sensitivity via Finite Differences

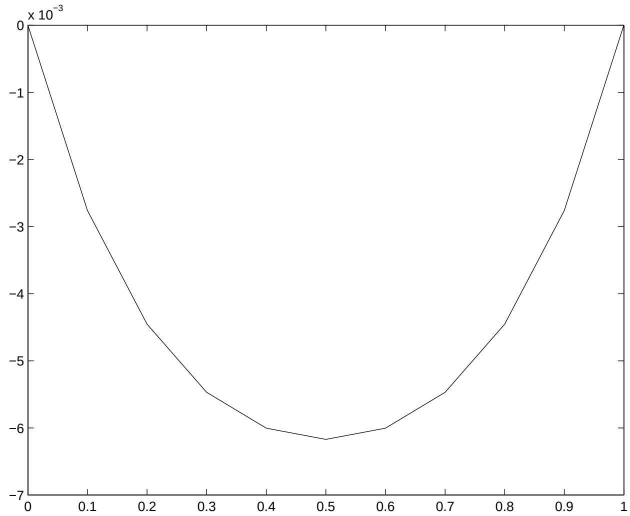

Figure 5. Sensitivity via Finite Differences

mum of $\| \mathbf{\Pi} \|$ corresponds to the zero of $\frac{\partial \| \mathbf{\Pi} \|}{\partial \gamma}$. Thus, the sensitivity produces derivatives that are consistent with the approximated design objective.

Next, we consider the norm of the feedback gain, $\| h \|$ as we vary γ (Figure 8). Again we see a nice quadratic function. As [2] conjectured, minimizing $\| \mathbf{\Pi} \|$ appears to be equivalent to maximizing $\| h \|$. Figure 9 depicts that the zero of $\frac{\partial \| h \|}{\partial \gamma}$ corresponds to the maximum of $\| h \|$.

8 Conclusions and Future Work

We have demonstrated the plausibility of using control performance measures to position actuators. For example, when attempting to control the heat over the interval [.7,.9], we see in Figure 6, that the optimum actuator location is near $\gamma = .7$. Presumably the in¤uence of the zero boundary condition at $x = 1$, leads to this result. In practical

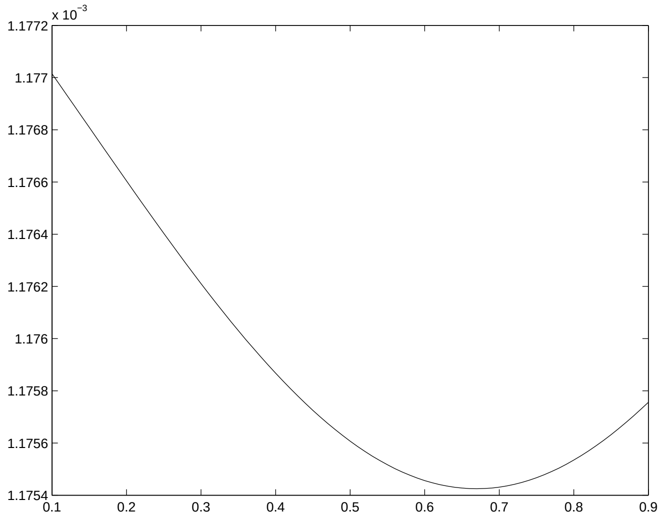

Figure 6. Norm of the Riccati Matrix

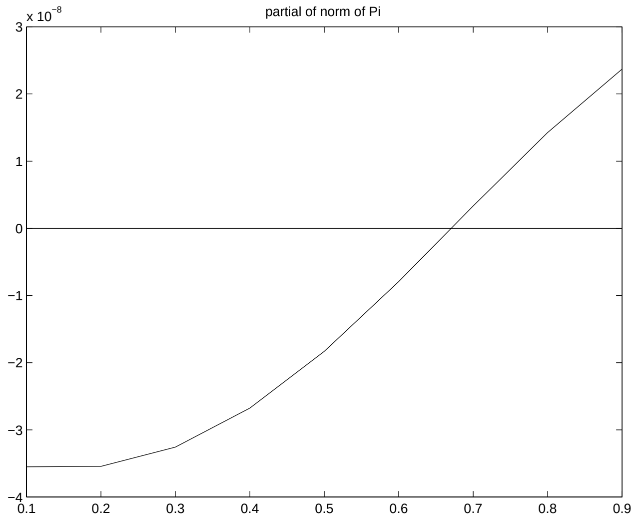

Figure 7. Partial of Norm of Riccati Matrix

problems, we don't have the luxury of computing the control performance objective over the entire parameter range, but rather need to resort to more systematic optimization algorithms. We have demonstrated the possibility of providing gradients to these algorithms using ef£cient sensitivity calculations.

In future work, we attempt to extend these results to higher dimensional problems, where the actuator location may be described by several parameters. These results are underway and are expected for the £nal version of this paper.

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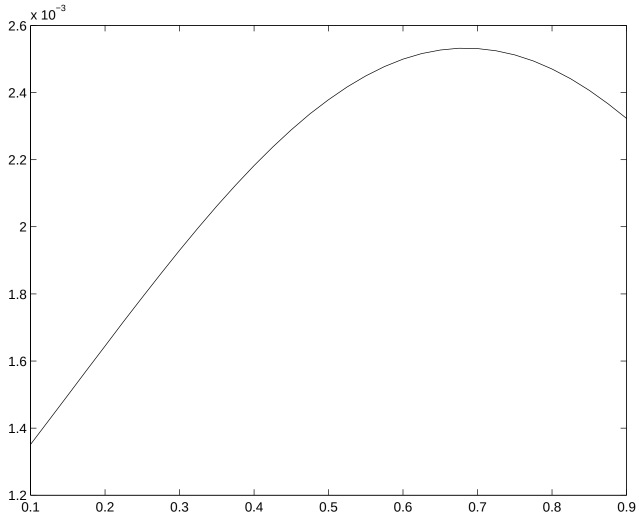

Figure 8. Norm of Functional Gain

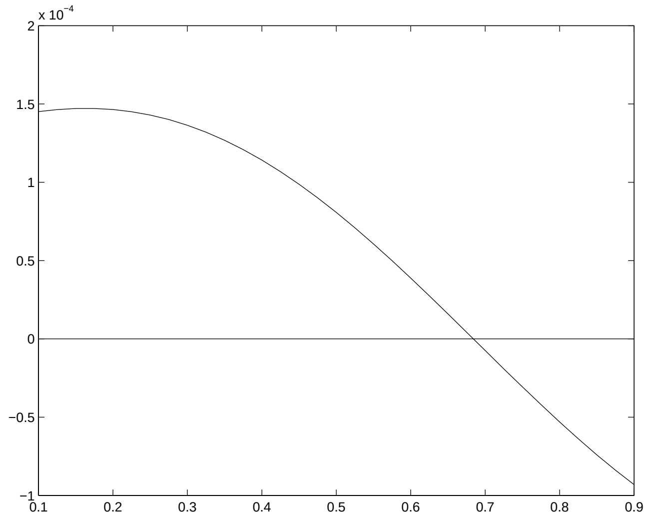

Figure 9. Partial of Norm of Functional Gain

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