Homework

1.

Each of the following augmented matrices $[A\vdots b]$ represents a linear system. How many equations and unkowns are represented in each of the systems? Identify whether each system is consistent or inconsistent. If the system is consistent, then use the `rref` command to determine a set of solutions.

(a) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 3 & -1 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & -1 & 1 & 3 \\ 1 & 0 & 1 & 4 \\ 3 & 1 & 1 & 5 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}$

(f) $\begin{bmatrix} 4 & -2 & 2 & 2 \\ 2 & 4 & -4 & 6 \\ 1 & 1 & 0 & 4 \end{bmatrix}$

What is the rank of the matrix A for each system in Problem 1?

Use the backslash command on each of the systems in Problem 1 after each one has been put in reduced row echelon form. If any case does not give you an answer, try removing any rows of zeros from the bottom and try it again.

2. The following problem deals with the condition number of a matrix. In general, a problem is considered ill-conditioned if small changes in parameters of the problem result in large changes in the solution of the problem. For a given problem in Matlab (which uses 17 digit precision), we can state the following about the error between an approximate solution, x, and the true solution, x^*:

$$\kappa(A)\times 10^{-17} \geq \frac{\|x - x^*\|}{\|x^*\|}$$

where $\kappa(A)$ is the condition number of the matrix A.

The relative error, $\dfrac{\|x - x^*\|}{\|x^*\|}$, can be used to find about how many digits of x and x^* will match. If the relative error is about 10^{-k}, then x and x^* will agree to about k places. This gives us a measure of how sensitive a solution may be to round off error.

The $(n \times n)$ Hilbert matrix which is defined by

$$A = \left[a_{i,j} \right]_{i,j=1}^{n} \text{ where } a_{i,j} = \frac{1}{i + j - 1}$$

is a good example of an ill-conditioned matrix. The $(n \times n)$ Hilbert matrix and its inverse can be calculated using the **hilb(n)** and **invhilb(n)** commands (the **invhilb** command will give the exact inverse if n is less than about 15).

a. Let H be the (6×6) Hilbert matrix. Solve the system $Hx = b$ where b is a vector of ones of appropriate size by using the backslash command.
b. What is the condition number of H. Use the **cond** command. Estimate the relative error in solving the system. How many digits do you expect to be correct in your answer?
c. Calculate the exact answer to part (a) using the **invhilb** command. That is, $x^* = H^{-1}b$. Find the relative error between x and x^*. Does the relative error agree with the accuracy you are getting in your answer?
d. Using the **cond** command, make an estimate of the largest system with an $(n \times n)$ Hilbert matrix that the backslash command can give a meaningful answer for (i.e. accuracy to at least two digits).
e. Verify your answer to part (d) by repeating parts (a) and (c) for your value of n.