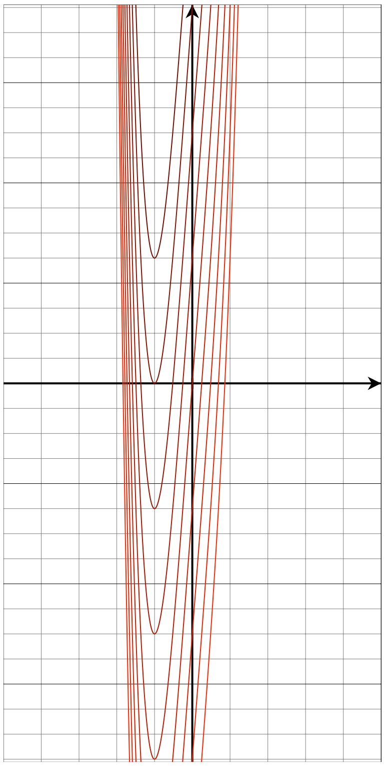
PROBLEM

Show that the equation $x^4 + 4x + c = 0$ has at most two real solutions.



Graphs of $f(x)$ for some values of c.

A look at graphs of $f(x) = x^4 + 4x + c$ for some values of c suggests that the equation may have zero, one, or two roots, but no more. We can use Rolle's theorem to prove this is in fact the case.

PROOF

Suppose that the equation has three or more real solutions. Label three of them x_1, x_2, and x_3 so that

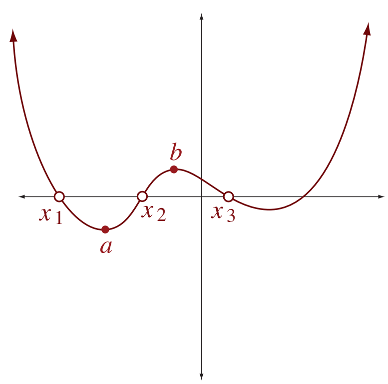
$$x_1 < x_2 < x_3.$$

Define a related function

$$f(x) = x^4 + 4x + c,$$

and note that solutions of the given equation are the zeros of this function: that is,

$$f(x_1) = f(x_2) = f(x_3) = 0.$$



The implications of Rolle's Theorem: if there are three zeros, there must be two critical points.

Since $f(x)$ is a polynomial, it is continuous and differentiable for all real numbers. Rolle's theorem applies, then, and according to Rolle's theorem there must be a number a between x_1 and x_2 for which $f'(a) = 0$. Similarly, there must be a number b between x_2 and x_3 for which $f'(b) = 0$.

Observe though:

$$\begin{aligned} f'(x) = 0: \quad & 4x^3 + 4 = 0 \\ & 4(x^3 + 1) = 0 \\ & x^3 + 1 = 0 \\ & x^3 = -1 \\ & x = -1. \end{aligned}$$

The derivative is only zero when $x = -1$. This falls short of the required two solutions. Therefore, the equation can have at most two real roots.