Math 2040 Test 1. Fall 2016

1. Find the equation of the secant line through the points $(0, f(0))$ and $(2, f(2))$ for the function $f(x) = x^2 + x + 1$.

2. An object's position as a function of time is given by the function $p(t) = -16t^2+60t$ (where t is measured in seconds and p is measured in feet). Find the object's average velocity between $t = 0$ and $t = 2$.

3. Evaluate each of the following limits. If the limit does not exist, say so.

(a) $\displaystyle\lim_{t\to 2} \frac{t^2-4}{t-2}$

(b) $\displaystyle\lim_{x\to 3} \frac{x-2}{x^2+2x+3}$

(c) $\displaystyle\lim_{h\to 0} \frac{\sqrt{2+h}-\sqrt{2}}{h}$

(d) $\displaystyle\lim_{h\to 0} \frac{(3+h)^{-1}-3^{-1}}{h}$

(e) $\displaystyle\lim_{x\to 5} \frac{x-5}{x^2-10x+25}$

(f) $\displaystyle\lim_{x\to 1} \cos\left[\pi\cdot\left(\frac{x-1}{x^4-1}\right)\right]$

4. Find the value(s) of c that make the function continuous:

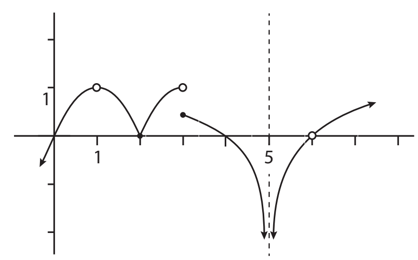
$$f(x) = \begin{cases} x^2+3c & \text{if } x < 0 \\ \cos x + c & \text{if } x \geq 0. \end{cases}$$

5. Use the intermediate value theorem to prove that the equation

$$x^4 + 5x + 1 = 0$$

has at least one solution in the interval $[-1, 1]$. Be sure to clearly explain your reasoning.

6. The graph of the function $f(x)$ is shown below.



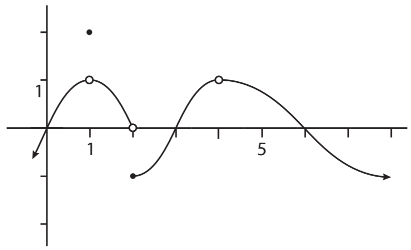
(a) For what value(s) c does $f(c)$ fail to exist?

(b) For what value(s) c is $f(c) = 0$?

(c) For what value(s) c does $\lim_{x\to c} f(x)$ fail to exist?

(d) For what value(s) c is $\lim_{x\to c} f(x) = 0$?

7. The graph of the function $f(x)$ is shown below. Identify all the points of discontinuity of $f(x)$. For each, explain (in a few words) why the function fails to be continuous there.



8. Give a $\delta - \epsilon$ argument to show that

$$\lim_{x\to 1} \frac{x^2-1}{2(x-1)} = 1.$$

SOLUTIONS

1. The slope of the secant line is

$$m = \frac{f(2)-f(0)}{2-0} = \frac{7-1}{2} = 3.$$

The point $(0, f(0)) = (0,1)$ is on the secant line. The equation of the secant line is $y = 3x+1$.

2. The average velocity is

$$v = \frac{p(2)-p(0)}{2-0} = \frac{56-0}{2-0} = 28\, ft/s.$$

3. (a)

$$\begin{aligned}
\lim_{t\to 2} \frac{t^2-4}{t-2} &= \lim_{t\to 2} \frac{(t-2)(t+2)}{t-2} \\
&= \lim_{t\to 2}(t+2) \\
&= 4.
\end{aligned}$$

(b)

$$\lim_{x\to 3} \frac{x-2}{x^2+2x+3} = \frac{3-2}{3^2+2\cdot 3+3} = \frac{1}{18}.$$

(c)

$$\begin{aligned}
&\lim_{h\to 0} \frac{\sqrt{2+h}-\sqrt{2}}{h} \\
&\quad = \lim_{h\to 0} \frac{\sqrt{2+h}-\sqrt{2}}{h} \cdot \frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}} \\
&\quad = \lim_{h\to 0} \frac{2+h-2}{\sqrt{2+h}+\sqrt{2}} \\
&\quad = \frac{1}{2\sqrt{2}}
\end{aligned}$$

(d)

$$\begin{aligned}
&\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{1}{3+h}-\dfrac{1}{3}}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{3}{3(3+h)}-\dfrac{3+h}{3(3+h)}}{h}\\
&= \lim_{h\to 0}\frac{-h}{3(3+h)}\cdot\frac{1}{h}\\
&= \lim_{h\to 0}\frac{-1}{3(3+h)}\\
&= -1/9.
\end{aligned}$$

(e)

$$\begin{aligned}
\lim_{x\to 5}\frac{x-5}{x^2-10x+25} &= \lim_{x\to 5}\frac{x-5}{(x-5)^2}\\
&= \lim_{x\to 5}\frac{1}{x-5}
\end{aligned}$$

The limit does not exist.

(f)

$$\begin{aligned}
&\lim_{x\to 1}\cos\left[\pi\cdot\left(\frac{x-1}{x^4-1}\right)\right]\\
&= \cos\left[\lim_{x\to 1}\frac{(x-1)\pi}{(x^2-1)(x^2+1)}\right]\\
&= \cos\left[\lim_{x\to 1}\frac{(x-1)\pi}{(x-1)(x+1)(x^2+1)}\right]\\
&= \cos\left[\lim_{x\to 1}\frac{\pi}{(x+1)(x^2+1)}\right]\\
&= \cos(\pi/4)\\
&= \sqrt{2}/2.
\end{aligned}$$

4. We need $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$. So

$$\begin{aligned}
\lim_{x\to 0^-}(x^2+3c) &= \lim_{x\to 0^+}(\cos x + c)\\
3c &= 1+c\\
c &= 1/2.
\end{aligned}$$

5. The function $f(x) = x^4+5x+1$ is a polynomial and so is continuous on $[-1,1]$. We see that

$$\begin{aligned}
f(-1) &= 1-5+1 = -3,\\
f(1) &= 1+5+1 = 7.
\end{aligned}$$

Since 0 is between -3 and 7, by the intermediate value theorem there is a number $c \in [-1,1]$ so that $f(c) = 0$. It is a solution to the equation.

6. (a) $c = 1,\ 5,\ 6$

(b) $c = 0,\ 2,\ 4$

(c) $c = 3,\ 5$

(d) $c = 0,\ 2,\ 4,\ 6$

7. There are three discontinuities: at $c = 1$, because $\lim_{x\to 1} f(x) \neq f(1)$; at $c = 2$, because $\lim_{x\to 2} f(x)$ does not exist; at $c = 4$, because $f(4)$ is undefined.

8. Given $\epsilon > 0$, we need to find $\delta > 0$ so that if $0 < |x-1| < \delta$, then

$$\left|\frac{x^2-1}{2(x-1)} - 1\right| < \epsilon.$$

Note that

$$\begin{aligned}
&\left|\frac{x^2-1}{2(x-1)} - 1\right| < \epsilon\\
\iff &\left|\frac{(x-1)(x+1)}{2(x-1)} - 1\right| < \epsilon\\
\iff &\left|\frac{x+1}{2} - 1\right| < \epsilon\\
\iff &\left|\frac{x-1}{2}\right| < \epsilon\\
\iff &\frac{|x-1|}{2} < \epsilon\\
\iff &|x-1| < 2\epsilon.
\end{aligned}$$

So, choose $\delta = 2\epsilon$. If $|x-1| < \delta$, then

$$\begin{aligned}
\left|\frac{x^2-1}{2(x-1)} - 1\right| &= \left|\frac{(x-1)(x+1)}{2(x-1)} - 1\right| \\
&= \left|\frac{x+1}{2} - 1\right| \\
&= \left|\frac{x-1}{2}\right| \\
&= \frac{|x-1|}{2} \\
&< \frac{2\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$