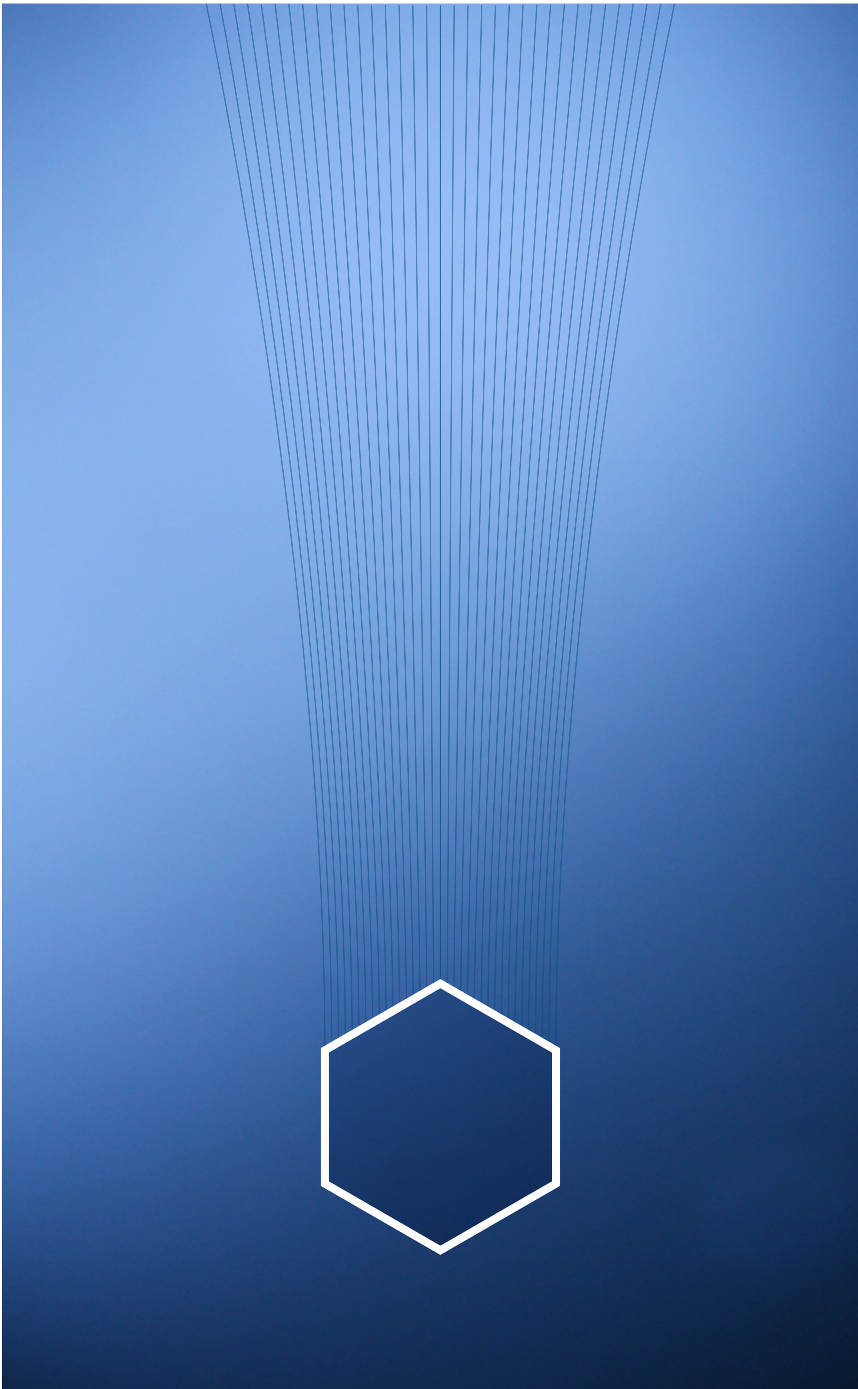


11 THE MANY SIDES OF **POLYGONS**

We have spent a lot of time talking about triangles, and I certainly do not want to give the impression that we are done with them, but in this lesson I would like to broaden the focus a little bit, and to look at polygons with more than three sides.

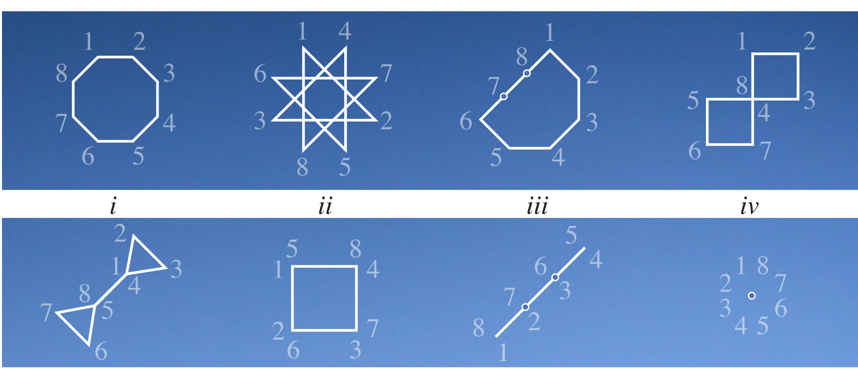
Definitions

Of course the first step is to get a working definition for the term *polygon*. This may not be as straightforward as you think. Remember the definition of a triangle? Three non-colinear points P_1, P_2, and P_3 defined a triangle. The triangle itself consisted of all the points on the segments P_1P_2, P_2P_3, and P_3P_1. At the very least, a definition of a polygon (as we think of them) involves a list of points and segments connecting each point to the next in the list, and then the last point back to the first:

The Vertices: $P_1, P_2, P_3, \ldots, P_n$

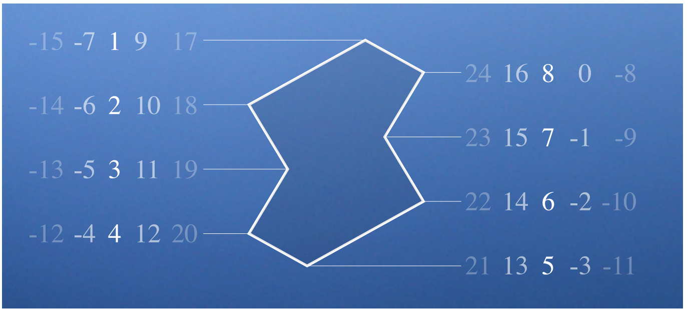
The Sides: $P_1P_2, P_2P_3, P_3P_4, \ldots, P_{n-1}P_n, P_nP_1$.

Now the one problem is this– what condition do you want to put on those points? With triangles, we insisted that the three points be non-colinear. What is the appropriate way to extend that beyond $n = 3$? This is not an easy question to answer. To give you an idea of some of the potential issues, let me draw a few configurations of points.



i *ii* *iii* *iv* *v* *vi* *vii* *viii*

Which of these do you think should be considered octagons (polygons with eight sides and eight vertices)?



Cyclic labeling of vertices

While you are mulling over that question, let me distract you by talking about notation. No matter what definition of polygon you end up using, your vertices will cycle around: $P_1, P_2, \ldots, P_n$ and then back to the start P_1. Because polygons do loop back around like this, sometimes you end up crossing from P_n back to P_1. For example, look at the listing of the sides of the polygon– all but one of them can be written in the form P_iP_{i+1}, but the last side, P_nP_1, doesn't fit that pattern. A proof involving the sides would have to go out of its way to be sure to mention that last side, and that is just not going to be very elegant. After all, other than the notation, the last side is not any different from the previous sides– it really should not need its own case. Fortunately, there is an easy way to sidestep this issue. What we can do is make our subscripts cycle just like the points do. Rather than using integer subscripts for the vertices, use integers modulo n (where n is the number of vertices). That way, for instance, in a polygon with eight vertices, P_9 and P_1 would stand for the same point since $9 \cong 1 \bmod 8$, and the sides of the polygon would be P_iP_{i+1} for $1 \leq i \leq 8$.

Now let's get back to the question of a definition. As I said at the start of the lesson, I think that there is still a spectrum of opinion on how a polygon should be defined. Some geometers (such as Grünbaum in *Are your polyhedra the same as my polyhedra* [2]) will tell you that any ordered listing of n points should define a polygon with n vertices and n sides. This includes listings where some or even all points are colinear or coinciding and can therefore can lead to some unexpected configurations: a six-sided polygon that appears to have only three sides, a triangle that looks like a line segment, a four-sided polygon that looks like a point. If you can get past the initial strangeness, though, there is definitely something to be said for this all-inclusive approach: for one thing, you never have to worry that moving points around would cause (for instance) your four-sided polygon to no longer be a four-sided polygon. This liberal definition would go something like this:

DEF: POLYGON (INCLUSIVE VERSION)
Any ordered list of points $\{P_i | 1 \leq i \leq n\}$ defines a polygon, written $P_1P_2 \cdots P_n$, with vertices P_i, $1 \leq i \leq n$, and sides P_iP_{i+1}, $1 \leq i \leq n$.

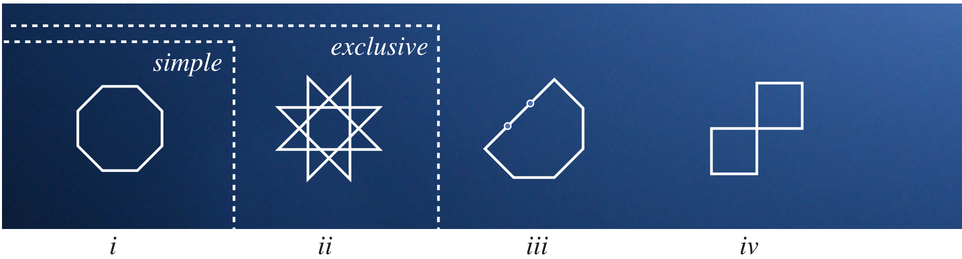
Other geometers like to put a few more restrictions on their polygons. I suspect that the most common objections to this all-inclusive definition would be:

(1) This collapsing of the vertices down to a single point or a single line as shown in illustrations (vii) and (viii) is unacceptable– polygons should have a two-dimensionality to them.

(2) The edges of a polygon should not trace back over one another as shown in illustrations (v) and (vi)– at most two edges should intersect each other once.

(3) On the topic of intersecting edges, only consecutive edges should meet at a vertex. Configurations such as the one shown in illustration (iv) do not define a single polygon, but rather several polygons joined together.

I don't know to what extent these added restrictions are historical conventions and to what extent they are truly fundamental to proving results on polygons. Let me point out though, that this all-inclusive definition doesn't quite work with our previous definition of a triangle: three colinear points would define a three-sided polygon, but not a triangle. Somehow, that just does not seem right. Were we to now to go back and liberalize our definition of a triangle to include these remaining three-sided polygons, it would cost us some theorems. For instance, neither A·S·A nor A·A·S would work in the case when all three vertices are colinear. So for that reason, let me also give a more restrictive definition of polygon that addresses the three concerns listed above.



DEF: POLYGON (EXCLUSIVE VERSION)
Any ordered list of points $\{P_i | 1 \leq i \leq n\}$ which satisfies the conditions

(1) no three consecutive points P_i, P_{i+1}, and P_{i+2} are colinear;
(2) if $i \neq j$, then P_iP_{i+1} and P_jP_{j+1} share at most one point;
(3) if $P_i = P_j$, then $i = j$;

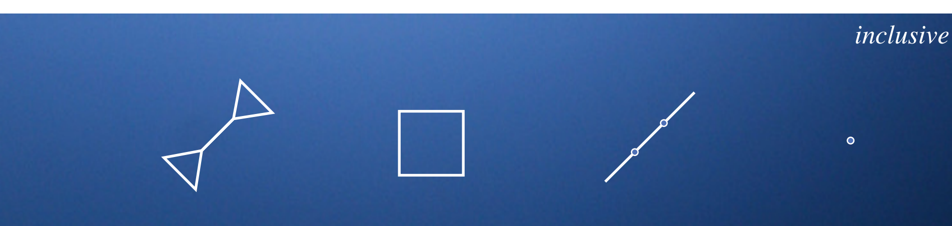
defines a polygon, written $P_1P_2 \cdots P_n$, with vertices P_i, $1 \leq i \leq n$, and sides P_iP_{i+1}, $1 \leq i \leq n$.

The crux of it is this: too liberal a definition and you are going to have to make exceptions and exclude degenerate cases; too conservative a definition and you end up short-changing your results by not expressing them at their fullest generality. After all of that, though, I have to say that I'm just not that worried about it, because for the most part, the polygons that we usually study are more specialized than either of those definitions– they are what are called *simple polygons*. You see, even in the more "exclusive" definition, the segments of a polygon are permitted to criss-cross one another. In a simple polygon, that type of behavior is not tolerated.

DEF: SIMPLE POLYGON
Any ordered list of points $\{P_i | 1 \leq i \leq n\}$ which satisfies the conditions

(1) no three consecutive points P_i, P_{i+1}, and P_{i+2} are colinear;
(2) if $i \neq j$ and P_iP_{i+1} intersects P_jP_{j+1} then either $i = j+1$ and the intersection is at $P_i = P_{j+1}$ or $j = i+1$ and the intersection is at $P_{i+1} = P_j$;

defines a simple polygon, written $P_1P_2 \cdots P_n$, with vertices P_i, $1 \leq i \leq n$, and sides P_iP_{i+1}, $1 \leq i \leq n$.

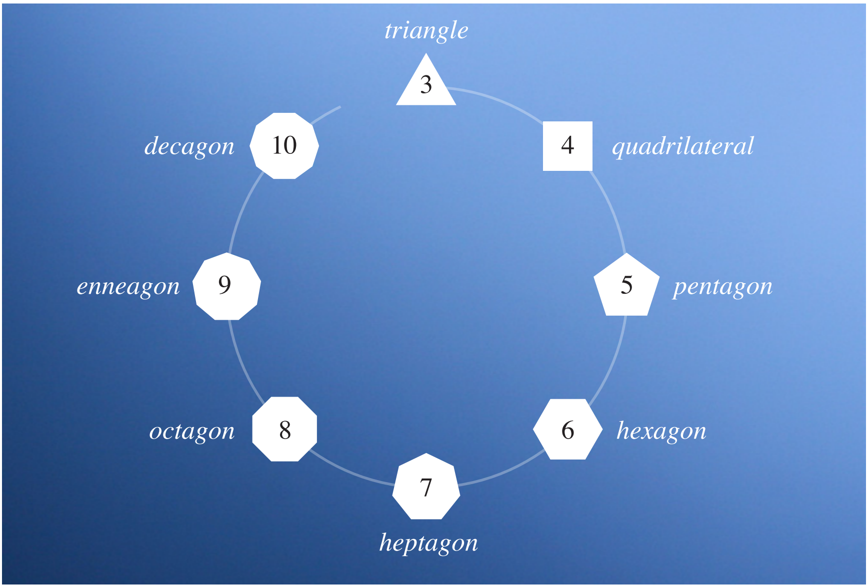


v *vi* *vii* *viii*

No matter how you choose to define a polygon, the definition of one important invariant of a polygon does not change:

DEF: PERIMETER
The perimeter P of a polygon is the sum of the lengths of its sides:

$$P = \sum_{i=1}^{n} |P_i P_{i+1}|.$$



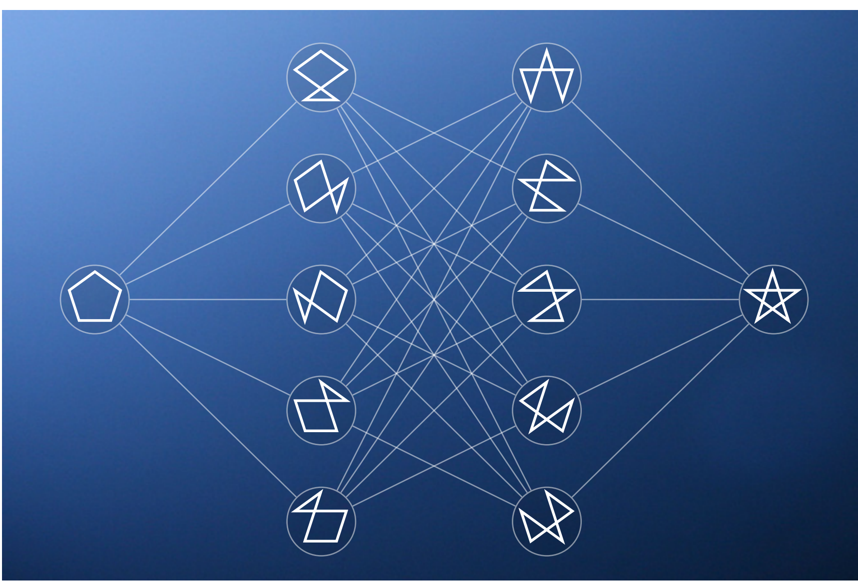
Names of polygons based upon the number of sides (and vertices).

Counting polygons

Two polygons are the same if they have the same vertices *and* the same edges. That means that the order that you list the vertices generally *does* matter– different orders can lead to different sets of sides. Not all rearrangements of the list lead to new polygons though. For instance, the listings $P_1P_2P_3P_4$ and $P_3P_4P_1P_2$ and $P_4P_3P_2P_1$ all define the same polygon: one with sides P_1P_2, P_2P_3, P_3P_4 and P_4P_1. More generally, any two listings which differ either by a cycling of the vertices or by a reversal of the order of one of those cyclings will describe the same polygon.

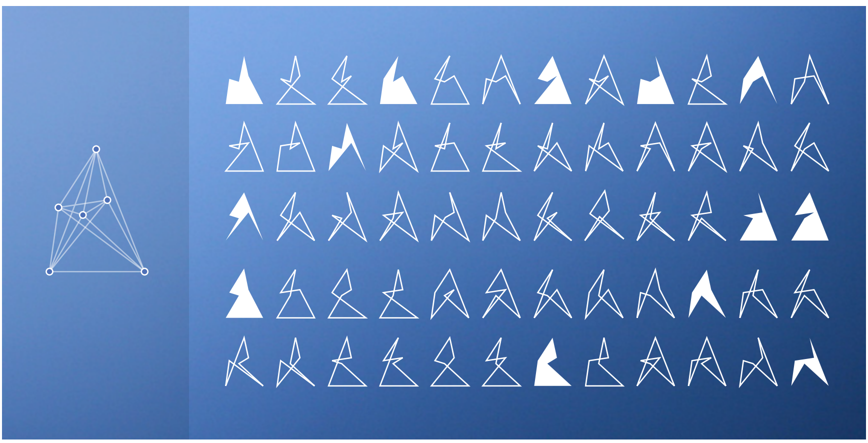
The 24 permutations of $1, 2, 3, 4$ *and the corresponding polygons on four points.*

So how many possible polygons are there on n points? That depends upon what definition of polygon you are using. The most inclusive definition of polygon leads to the easiest calculation, for in that case, any configuration on n points results in a polygon. As you probably know from either probability or group theory, there are $n!$ possible orderings of n distinct elements. However for each such list there are n cyclings of the entries and n reversals of those cyclings, leading to a total of $2n$ listings which all correspond to the same polygon. Therefore, there are $n!/(2n) = (n-1)!/2$ possible polygons that can be built on n vertices. Notice that when $n = 3$, there is only one possibility, and that is why none of this was an issue when we were dealing with triangles.



The 12 polygons on a configuration of five points. In this illustration, segments connect two polygons which differ by a swap of two adjacent vertices.

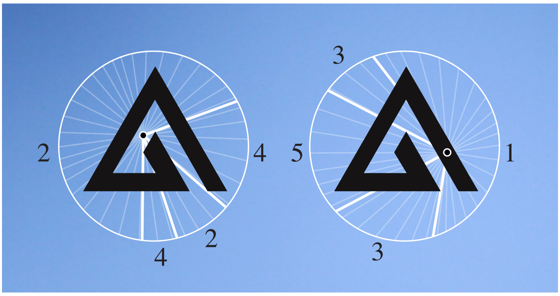
If instead you are using the more exclusive definition of a polygon, then things are a bit more complicated. If the vertices are in "general position" so that any combination of segments P_iP_j satisfies the requirements outlined in that definition, then there are just as many exclusive polygons as inclusive polygons: $(n-1)!/2$. Probabilistically, it is most likely that any n points will be in such a general position, but it is also true that as n grows, the number of conditions required to attain this general position increases quite rapidly. Even less understood is the situation for simple polygons. The condition of simplicity throws the problem from the relatively comfortable world of combinatorics into a much murkier geometric realm.



Thirteen of the sixty polygons on this configuration of six points are simple.

Interiors and exteriors

One characteristic of the triangle is that it chops the rest of the plane into two sets, an interior and an exterior. It isn't so clear how to do that with a polygon (this is particularly true if you are using the inclusive definition of the tem, but to a lesser extent is still true with the exclusive definition). Simple polygons, though, do separate the plane into interior and exterior. This is in fact a special case of the celebrated Jordan Curve Theorem, which states that every simple closed curve in the plane separates the plane into an interior and an exterior. The Jordan Curve Theorem is one of those notorious results that seems like you could knock out in an afternoon, but is actually brutally difficult. In the special case of simple polygons, our case, there are simpler proofs. I am going to describe the idea behind one such proof from *What is Mathematics?* by Courant and Robbins [1].



Rays from a point. The number of intersections with a polygon (in black) depends upon which ray is chosen, but the parity (even or odd) does not.

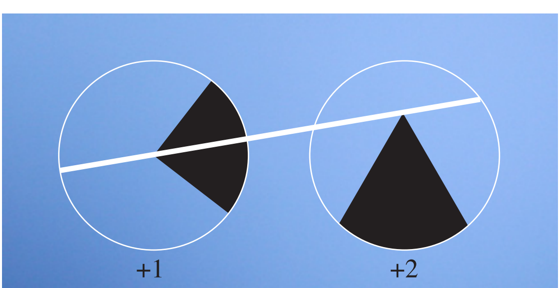
THM: POLYGONAL PLANE SEPARATION

Every simple polygon separates the remaining points of the plane into two connected regions.

Proof. Let $\mathscr{P}$ be a simple polygon, and let p be a point which is not on $\mathscr{P}$. Now let's look at a ray R_p whose endpoint is p. As long as R_p does not run exactly along an edge, it will intersect the edges of $\mathscr{P}$ a finite number of times (perhaps none). You want to think of each such intersection as a crossing of R_p into or out of $\mathscr{P}$.

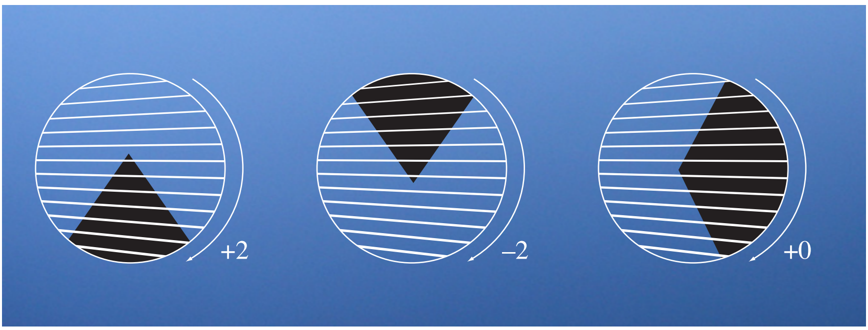
Since there are only finitely many intersections, they are all within a finite distance of P. That means that eventually R_p will pass beyond all the points of $\mathscr{P}$. This is the essence of this argument: eventually the ray is outside of the polygon, so by counting back the intersections crossing into and out of the polygon, we can figure out whether the beginning of the ray, P is inside or outside of $\mathscr{P}$. The one situation where we have to be a little careful is when R_p intersects a vertex of $\mathscr{P}$. Here is the way to count those intersections:

$$\begin{cases} \text{once if } R_p \text{ separates the two neighboring edges;} \\ \text{twice if } R_p \text{ does not separate them.} \end{cases}$$



Procedure for counting intersections at a vertex.

Now when you count intesections this way, the number of intersections depends not just upon the point p, but also upon the of direction of R_p. The key, though, is that there is one thing which does not depend upon the direction– whether the number of intersections is odd or even, the "parity" of p. To see why, you have to look at what happens as you move the ray R_p around, and in particular what causes the number of intersections to change. Without giving overly detailed explanation, changes can



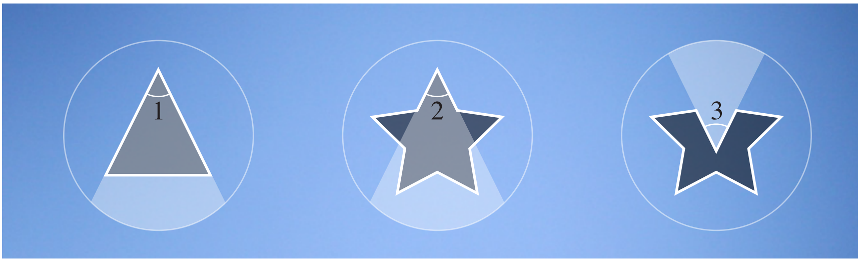
As the ray shifts across a vertex, the intersection count changes by +2, -2, *or* 0, *all even numbers*.

only happen when R_p crosses one of the vertices of $\mathscr{P}$. Each such vertex crossing corresponds to either an increase in 2 in the number of crossings, a decrease by 2 in the number of crossings, or no change in the number of crossings. In each case, the parity is not changed. Therefore $\mathscr{P}$ separates the remaining points of the plane into two sets– those with even parity and those with odd parity. Furthermore, each of those sets is connected in the sense that by tracing just to one side of the edges of $\mathscr{P}$, it is possible to lay out a path of line segments connecting any two points with even parity, or any two points with odd parity. □

DEF: POLYGON INTERIOR AND EXTERIOR

For any simple polygon $\mathscr{P}$, the set of points with odd parity (as described in the last proof) is called the *interior* of $\mathscr{P}$. The set of points with even parity is called the *exterior* of $\mathscr{P}$.

I will leave it to you to prove the intuitively clear result: that a polygon's interior is always a bounded region and that its exterior is always an unbounded region.

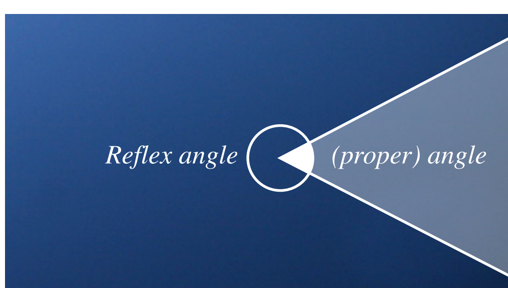


Angle interiors and polygon interiors.

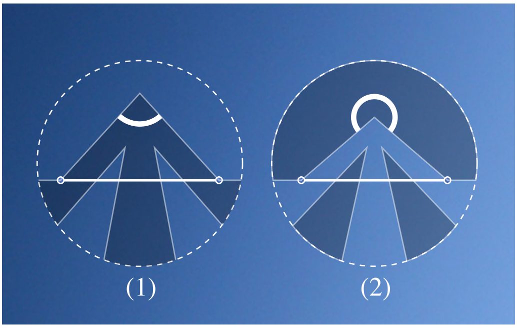
Interior angles: two dilemmas

Now I want to talk a little bit about the interior angles of a simple polygon. If you would, please look at the three marked angles in the polygons above. The first, $\angle 1$ is the interior angle of a triangle. You can see that the entire interior of the triangle is contained in the interior of the angle, and that seems proper, that close connection between the interiors of the interior angles and the interior of the triangle. Now look at $\angle 2$, and you can see that for a general simple polygon, things do not work quite as well: the entire polygon does not lie in the interior of this angle. But at least the part of the polygon interior which is closest to that vertex is in the interior of that angle. Finally look at $\angle 3$: the interior of $\angle 3$ encompasses exactly none of the interior of the polygon– it is actually pointing away from the polygon.

Let me address the issue surrounding $\angle 3$ first. We have said that two non-opposite rays define a single angle, and later established a measure for that angle– some number between 0 and 180°. But really, two rays like this divide the plane into *two* regions, and correspondingly, they should form *two* angles. One is the proper angle which we have already dealt with. The other angle is what is called a *reflex angle*. Together, the measures of the proper angle and the reflex angle formed by any two rays should add up to



A pair of explementary angles.



The dark region shows the polygon interior around a vertex. In (1), the connecting segment begins in the interior, so the interior angle is the proper angle. In (2), the connecting segment begins in the exterior, so the interior angle is the reflex angle.

$360°$. There does not seem to be a standard bit of terminology to describe this relationship between angles; I have seen the term "conjugate" as well as the term "explementary". So the problem with $\angle 3$ is that the interior angle isn't the proper angle, but instead, that it is its explement.

Now as long as the polygon is fairly simple (no pun intended) this is all fairly clear, but suppose that we were looking at an angle $\angle P_i$ in a much more elaborate polygon. Should the interior angle at P_i be the proper angle $\angle P_{i-1}P_iP_{i+1}$ or its conjugate? Well, to answer that question, you need to look at the segment $P_{i-1}P_{i+1}$. It may cross into and out of the interior of the polygon, but if the interior angle is the proper angle, then the first and last points of $P_{i-1}P_{i+1}$ (the ones closest to P_{i-1} and P_{i+1}) will be in the interior of the polygon. If the interior angle is the reflex angle, then the first and last points of $P_{i-1}P_{i+1}$ won't be in the interior of the polygon.

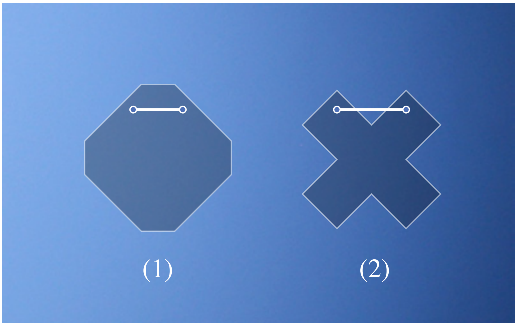
With the interior angles of a polygon now properly accounted for, we can define what it means for two polygons to be congruent.

DEF: POLYGON CONGRUENCE

Two polygons $\mathscr{P} = P_1P_2\cdots P_n$ and $\mathscr{Q} = Q_1Q_2\cdots Q_n$ are congruent, written $\mathscr{P} \simeq \mathscr{Q}$ if all corresponding sides and interior angles are congruent:

$$P_iP_{i+1} \simeq Q_iQ_{i+1} \quad \& \quad \angle P_i \simeq \angle Q_i, \text{ for all } i.$$

Now let's take a look at $\angle 2$, where not all of the interior of the polygon lies in the interior of the angle. The problem here is a little bit more intrinsic– I don't think you are going to be able to get around this one by fiddling with definitions (well, not at least without making a lot of questionable compromises). There is, though, a class of simple polygon for which the polygon interior always lies in the interior of each interior angle. These are the convex polygons.



A (1) convex and (2) a non-convex polygon. In the second, a segment joins two points in the interior, but passes outside of the polygon.

DEF: CONVEX POLYGON

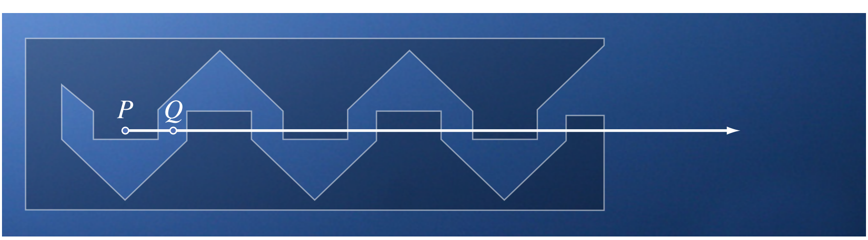
A polygon $\mathscr{P}$ is *convex* if, for any two points p and q in the interior of $\mathscr{P}$, the entire line segment pq is in the interior of $\mathscr{P}$.

Convexity is a big word in geometry and it comes up in a wide variety of contexts. Our treatment here will be very elementary, and just touch on the most basic properties of a convex polygon.

THM: CONVEXITY 1

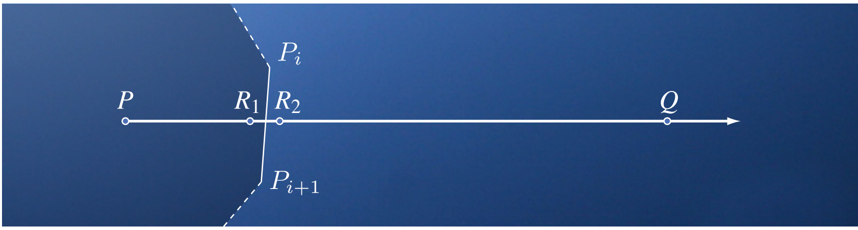
If $\mathscr{P} = P_1P_2\cdots P_n$ is a convex polygon, then all the points of the interior of $\mathscr{P}$ lie on the same side of each of the lines P_iP_{i+1}.

Proof. The fundamental mechanism that makes this proof work is the way that we defined the interior and exterior of a polygon by drawing a ray out and counting how many times it intersects the sides of $\mathscr{P}$. Suppose that P and Q lie on opposite sides of a segment P_iP_{i+1}, so that PQ intersects P_iP_{i+1}. Suppose further that PQ intersects no other sides of the polygon. Then the ray $PQ\rightarrow$ will intersect $\mathscr{P}$ one more time than the ray $(QP\rightarrow)^{op}$. Therefore P and Q will have different parities, and so one of P and Q will be an interior point and the other an exterior point.



A single side of the polygon comes between P and Q– one must be outside and one must be inside.

Now on to the proof, a proof by contradiction. Suppose that both P and Q are in the interior of a convex polygon, but that they are on the opposite sides of $\leftarrow P_iP_{i+1}\rightarrow$. After the previous discussion, it is tempting to draw a picture that looks like

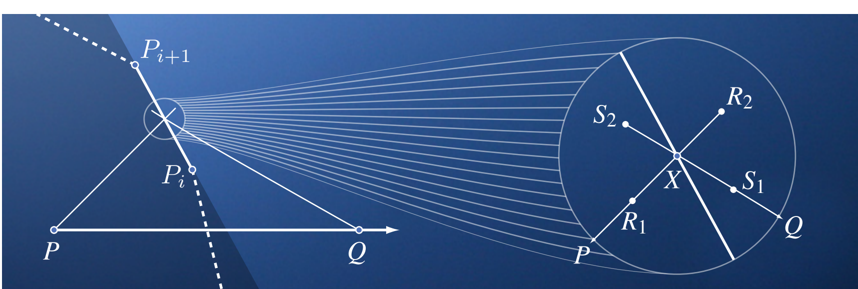


In that case, only one of R_1, R_2 can be in the interior of $\mathscr{P}$ and so $\mathscr{P}$ can't be convex and we have our contradiction. But that misses an important (and indeed likely) scenario– the one in which PQ intersects the line $\leftarrow P_iP_{i+1}\rightarrow$ but not the segment P_iP_{i+1}. To deal with that scenario, we are going to have to maneuver the intersection so that it does occur on the segment, which requires a bit more delicate argument.

Choose a point X which is between P_i and P_{i+1}. We will relay the interior/exterior information from P and Q back to points which are in close proximity to X. Choose points R_1, R_2, S_1 and S_2 so that

$$P * R_1 * X * R_2 \quad Q * S_1 * X * S_2.$$

In addition, we want to make sure that these points are so close together that none of the other sides of $\mathscr{P}$ get in the way, so we will require (1) R_1S_1 does intersect the side P_iP_{i+1}, but that (2) none of the edges other than P_iP_{i+1} comes between any two of these points. A polygon only has finitely many edges, so yes, it is possible to do this. Then R_1 and R_2 lie on different sides of the segment P_iP_{i+1}, so one is in the interior and one



is in the exterior. Suppose that R_2 is the interior point. Then, since $\mathscr{P}$ is convex, and R_1 is between two interior points P and R_2, R_1 must also be an interior point. Since R_1 and R_2 cannot both be interior points, that means that R_1 is the interior point. Applying a similar argument to Q, S_1 and S_2, you can show that S_1 must also be an interior point. But now R_1 and S_1 are on opposite sides of P_iP_{i+1}, so they cannot both be interior points. This is the contradiction. □

There are a couple immediate corollaries of this– I am going to leave the proofs of both of these to you.

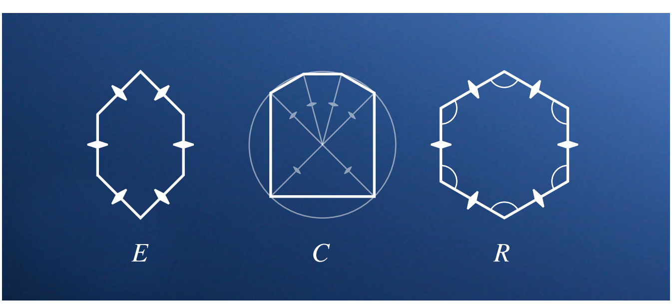
THM: CONVEXITY 2
If $\mathscr{P}$ is a convex polygon, then the interior of $\mathscr{P}$ lies in the interior of each interior angle $\angle P_i$.

THM: CONVEXITY 3
If $\mathscr{P}$ is a convex polygon, then each of its interior angles is a proper angle, not a reflex angle.

Polygons of note

To finish this chapter, I want to mention a few particularly well-behaved types of polygons.

TYPES OF POLYGONS
An *equilateral* polygon is one in which all sides are congruent. A *cyclic* polygon is one in which all vertices are equidistant from a fixed point (hence, all vertices lie on a circle, to be discussed later). A *regular* polygon is one in which all sides are congruent and all angles are congruent.

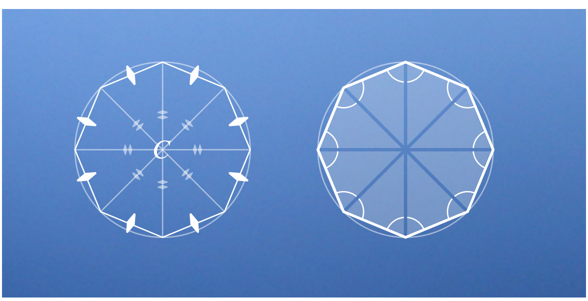


E: equilateral
C: cyclic
R: regular

The third of these types is actually a combination of the previous two types as the next theorem shows.

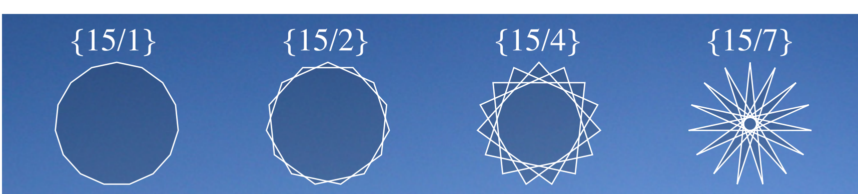
THM: EQUILATERAL + CYCLIC
A polygon $\mathscr{P}$ which is both equilateral and cyclic is regular.

Proof. We need to show that the interior angles of $\mathscr{P}$ are all congruent. Let C be the point which is equidistant from all points of $\mathscr{P}$. Divide $\mathscr{P}$ into a set of triangles by constructing segments from each vertex to C. For any of these triangles, we wish to distinguish the angle at C, the central angle, from the other two angles in the triangle. Note that the two constructed sides of these triangles are congruent. By the Isosceles Triangle Theorem, the two non-central angles are congruent. As well, by S·S·S, all of these triangles are congruent to each other. In particular, all non-central angles of all the triangles are congruent. Since adjacent pairs of such angles comprise an interior angle of $\mathscr{P}$, the interior angles of $\mathscr{P}$ are congruent. □



Because of S-S-S and the Isosceles Triangle Theorem, polygons which are equilateral and cyclic are regular.

While we normally think of regular polygons as I have shown them above, there is nothing in the definition that requires a regular polygon to be simple. In fact, there are non-simple regular polygons– such a polygon is called a *star polygon*.



There is a regular star n-gon for each integer p between 1 *and n/*2 *that is relatively prime to n. Shown here: n*=15*. The* {*n/p*} *notation is called the Schläfli symbol.*

Exercises

1. Verify that a triangle is a convex polygon.
2. A diagonal of a polygon is a segment connecting nonadjacent vertices. How many diagonals does an n-gon have?
3. Prove theorems 2 and 3 on convexity.
4. Prove that a regular convex polygon is cyclic (to find that equidistant point, you may have to consider the odd and even cases separately).
5. Prove that if a polygon is convex, then all of its diagonals lie entirely in the interior of the polygon (except for the endpoints).
6. Prove that if a polygon is not convex, then at least one of its diagonals does not lie entirely in the interior of the polygon.
7. Verify that the perimeter of any polygon is more than twice the length of its longest side.
8. Prove that the sum of the interior angles of a convex n-gon is at most $180^\circ(n-2)$.
9. Prove that if a polygon $\mathscr{P}$ is convex, then there are no other simple polygons on that configuration of vertices.

References

[1] Richard Courant and Herbert Robbins. *What is Mathematics? : an elementary approach to ideas and methods*. Oxford University Press, London, 1st edition, 1941.

[2] Branko Grünbaum. Are your polyhedra the same as my polyhedra? *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, 2003.