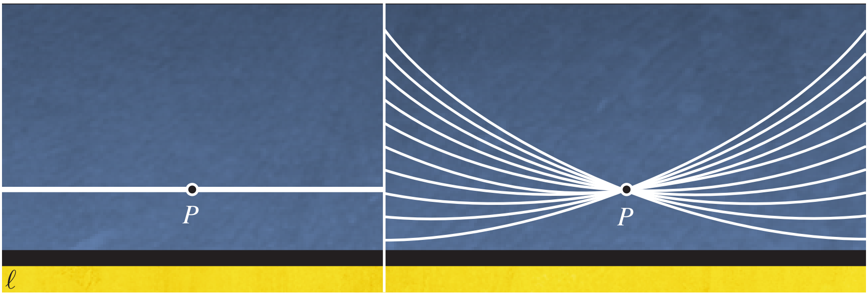
2 EUCLIDEAN GEOMETRY

My goal with all of these lessons is to provide an introduction to both Euclidean non-Euclidean geometry. The two geometries share many features, but they also have very fundamental and radical differences. Neutral geometry is the part of the path they have in common and that is what we have been studying so far, but I think we have finally come to the fork in the path. That fork comes when you try to answer this question:

> Given a line ℓ and a point P which is not on ℓ, how many lines pass through P and are parallel to ℓ?

Using just the axioms of neutral geometry, you can prove that there is always at least one such parallel. You can also prove that if there is more than one parallel, then there must be infinitely many. But that is the extent of what the neutral axioms can say. The neutral axioms just aren't enough to determine whether there is one parallel or many. This is what separates Euclidean and non-Euclidean geometry– a single axiom: the final axiom of Euclidean geometry calls for *exactly one* parallel, the final axiom of non-Euclidean geometry calls for *more than one* parallel.

13 REGARDING PARALLELS,
A DECISION IS MADE



Euclidean parallel *non-Euclidean parallels*

The next several lessons are devoted to Euclidean geometry. Now you have to remember that Euclidean geometry is several millenia old, so there is a lot of it. All that I hope to do in these lessons is to cover the fundamentals, but there are many excellent books that do much more. *Geometry Revisited* [1] by Coxeter and Greitzer is an excellent one.

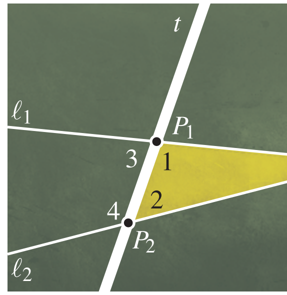
The first order of business is to put that final axiom in place. There are many formulations of the parallel axiom for Euclidean geometry, but the one that I think gets right to the heart of the matter is Playfair's Axiom, named after the Scottish mathematician John Playfair.

PLAYFAIR'S AXIOM
Let ℓ be a line, and let P be a point which is not on ℓ. Then there is exactly one line through P which is parallel to ℓ.

In this lesson I would like to look at a small collection of theorems which are almost immediate consequences of this axiom, and as such, are at the very core of Euclidean geometry. The first of these is Euclid's Fifth Postulate. This is the controversial postulate in *The Elements*, but also the one that guarantees the same parallel behavior that Playfair's Axiom provides. In my opinion, Euclid's postulate is a little unwieldy, particularly when compared to Playfair's Axiom, but it is the historical impetus for so much of what followed. So let's use Playfair's Axiom to prove Euclid's Fifth Postulate.

EUCLID'S FIFTH POSTULATE
If lines ℓ_1 and ℓ_2 are crossed by a transversal t, and the sum of adjacent interior angles on one side of t measure less than 180°, then ℓ_1 and ℓ_2 intersect on that side of t.

Proof. First, some labels. Start with lines ℓ_1 and ℓ_2 crossed by transversal t. Label P_1 and P_2, the points of intersection of t with ℓ_1 and ℓ_2 respectively. On one side of t, the two adjacent interior angles should add up to less than 180°. Label the one at P_1 as $\angle 1$ and the one at P_2 at $\angle 2$. Label the supplement of $\angle 1$ as $\angle 3$ and label the supplement of $\angle 2$ as $\angle 4$.

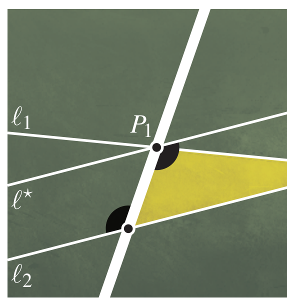


The labels.

Primarily, of course, this postulate is about the location of the intersection of ℓ_1 and ℓ_2. But you don't want to overlook an important prerequisite: the postulate is also guaranteeing that ℓ_1 and ℓ_2 *do* intersect. That's really the first thing we need to show. Note that $\angle 1$ and $\angle 4$ are alternate interior angles, but they are not congruent– if they were, their supplements $\angle 2$ and $\angle 3$ would be too, and then

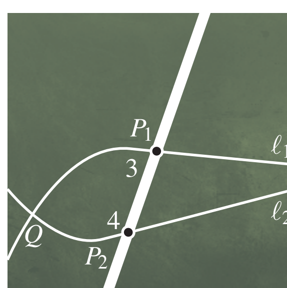
$$(\angle 1) + (\angle 2) = (\angle 1) + (\angle 3) = 180°.$$

There is, however, another line $\ell^\star$ through P_1 which does form an angle congruent to $\angle 4$ (because of the Angle Construction Postulate), and by the Alternate Interior Angle Theorem, $\ell^\star$ must be parallel to ℓ_2. Because of Playfair's Axiom, $\ell^\star$ is the only parallel to ℓ_2 through P_1. That means ℓ_1 intersects ℓ_2.

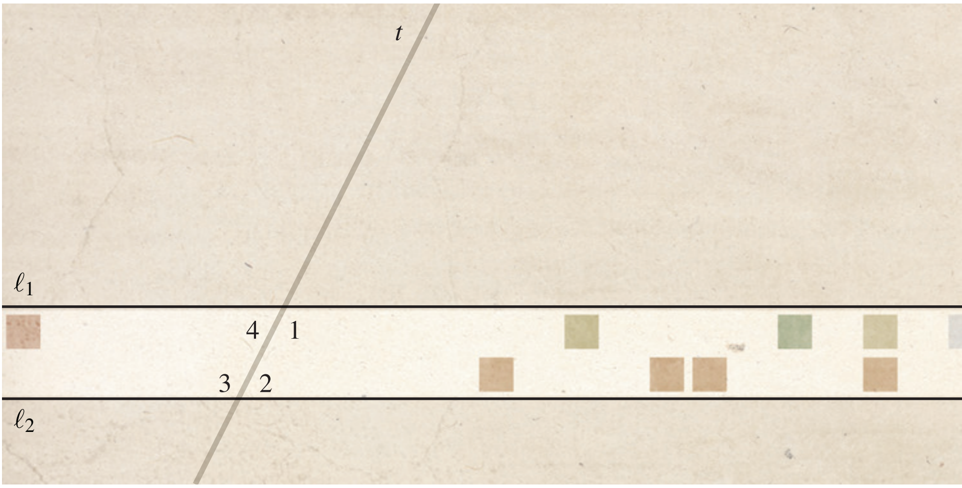


Constructing the unique parallel.

The second part of the proof is to figure out on which side of t that ℓ_1 and ℓ_2 cross. Let's see what would happen if they intersected at a point Q on the wrong side of t: the side with $\angle 3$ and $\angle 4$. Then the triangle $\triangle P_1P_2Q$ would have two interior angles, $\angle 3$ and $\angle 4$, which add up to more than 180°. This violates the Saccheri-Legendre theorem. So ℓ_1 and ℓ_2 cannot intersect on the side of t with $\angle 3$ and $\angle 4$ and that means that they must intersect on the side with $\angle 1$ and $\angle 2$. □



An impossible triangle on the wrong side of t.



One of the truly useful theorems of neutral geometry is the Alternate Interior Angle Theorem. In fact, we just used it in the last proof. But you may recall from high school geometry, that the converse of that theorem is often even more useful. The problem is that the converse of the Alternate Interior Angle Theorem can't be proved using just the axioms of neutral geometry. It depends upon Euclidean behavior of parallel lines.

CONVERSE OF THE ALTERNATE INTERIOR ANGLE THEOREM
If ℓ_1 and ℓ_2 are parallel, then the pairs of alternate interior angles formed by a transversal t are congruent.

Proof. Consider two parallel lines crossed by a transversal. Label adjacent interior angles: $\angle 1$ and $\angle 2$, and $\angle 3$ and $\angle 4$, so that $\angle 1$ and $\angle 4$ are supplementary and $\angle 2$ and $\angle 3$ are supplementary. That means that the pairs of alternate interior angles are $\angle 1$ and $\angle 3$ and $\angle 2$ and $\angle 4$. Now, we just have to do a little arithmetic. From the two pairs of supplementary angles:

$$\begin{cases} (\angle 1) + (\angle 4) = 180^\circ & (i) \\ (\angle 2) + (\angle 3) = 180^\circ. & (ii) \end{cases}$$

Notice that if you add all four angles together, then

$$(\angle 1) + (\angle 2) + (\angle 3) + (\angle 4) = 360^\circ.$$

Now, here is where Euclid's Fifth comes into play– and actually, we will need to use the contrapositive. You see, ℓ_1 and ℓ_2 are parallel, and that means that they do not intersect on either side of t. Therefore Euclid's Fifth says that on neither side of t may the sum of adjacent interior angles be less than 180°:

$$\begin{cases} (\angle 1) + (\angle 2) \geq 180^\circ \\ (\angle 3) + (\angle 4) \geq 180^\circ. \end{cases}$$

If either one of these sums was greater than 180°, though, the sum of all four angles would have to be more than 360°– we saw above that is not the case, so the inequalities are actually equalities:

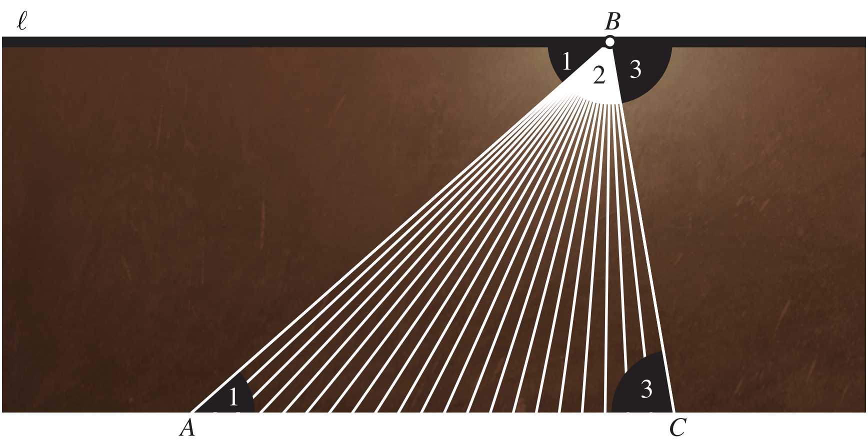
$$\begin{cases} (\angle 1) + (\angle 2) = 180^\circ & (iii) \\ (\angle 3) + (\angle 4) = 180^\circ. & (iv) \end{cases}$$

Now you have two systems of equations with four unknowns– it is basic algebra from here. Subtract equation (iv) from equation (i) to get $(\angle 1) = (\angle 3)$. Subtract equation (iii) from equation (i) to get $(\angle 2) = (\angle 4)$. The alternate interior angles are congruent. $\square$

One of the key theorems we proved in the neutral geometry section was the Saccheri-Legendre Theorem: that the angle sum of a triangle is at most 180°. That's all you can say with the axioms of neutral geometry, but in a world with Playfair's Axiom and the converse of the Alterante Interior Angle Theorem, there can be only one triangle angle sum.

THM
The angle sum of a triangle is 180°.

Proof. Consider a triangle $\triangle ABC$. By Playfair's Axiom, there is a unique line ℓ through B which is parallel to $\leftarrow AC \rightarrow$. That line and the rays $BA\rightarrow$ and $BC\rightarrow$ form three angles, $\angle 1$, $\angle 2$ and $\angle 3$ as I have shown in the illustration below.



By the converse of the Alternate Interior Angle Theorem, two pairs of alternate interior angles are congruent:

$$\angle 1 \simeq \angle A \quad \angle 3 \simeq \angle C.$$

Therefore, the angle sum of $\triangle ABC$ is

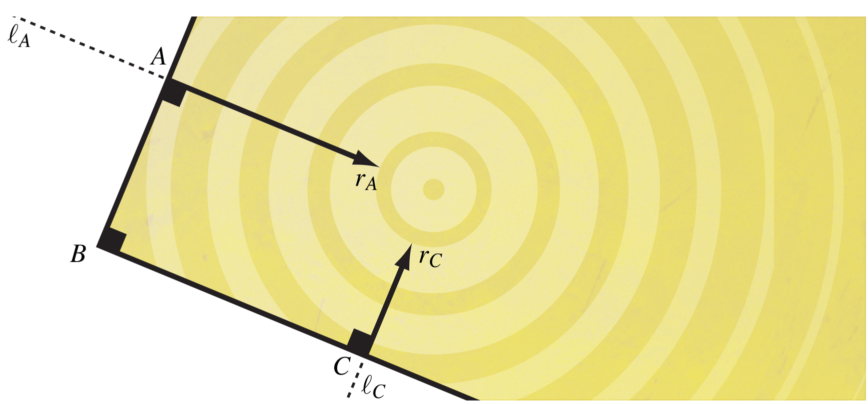
$$\begin{aligned} s(\triangle ABC) &= (\angle A) + (\angle B) + (\angle C) \\ &= (\angle 1) + (\angle 2) + (\angle 3) \\ &= 180^\circ. \end{aligned}$$

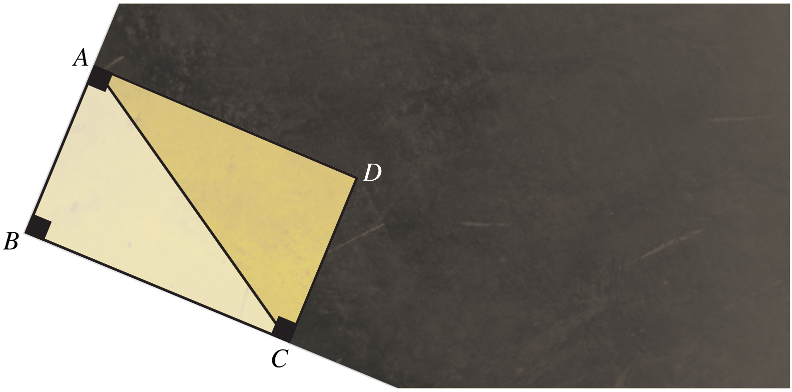
$\square$

In the last lesson on quadrilaterals I talked a little bit about the uncertain status of rectangles in neutral geometry– that it is pretty easy to make a convex quadrilateral with three right angles, but that once you have done that, there is no guarantee that the fourth angle will be a right angle. Here it is now in the Euclidean context:

RECTANGLES EXIST
Let $\angle ABC$ be a right angle. Let r_A and r_B be rays so that: r_A has endpoint A, is on the same side of $\leftarrow AB \rightarrow$ as C, and is perpendicular to $\leftarrow AB \rightarrow$; r_C has endpoint C, is on the same side of $\leftarrow BC \rightarrow$ as A, and is perpendicular to $\leftarrow BC \rightarrow$. Then r_A and r_C intersect at a point D, and the angle fomed at this intersection, $\angle ADC$, is a right angle. Therefore $\square ABCD$ is a rectangle.

Proof. The first bit of business is to make sure that r_A and r_C intersect. Let ℓ_A and ℓ_C be the lines containing r_A and r_C respectively. By the Alternate Interior Angle Theorem, the right angles at A and B mean that ℓ_A and $\leftarrow BC \rightarrow$ are parallel. So $\leftarrow BC \rightarrow$ is the one line parallel to ℓ_A through C, and that means that ℓ_C cannot be parallel to ℓ_A: it has to intersect ℓ_A. Let's call that point of intersection D. Now in the statement of the theorem, I claim that it is the *rays*, not the lines, that intersect. That means that we need to rule out the possibility that the intersection of ℓ_A and ℓ_C might happen on one (or both) of the opposite rays. Observe that since ℓ_A is parallel to $\leftarrow BC \rightarrow$, all of the points of ℓ_A are on the same side of $\leftarrow BC \rightarrow$ as A. None of the points of r_C^{op} are on that side of BC, so D cannot be on r_C^{op}. Likewise, all the points of ℓ_C are on the same side of $\leftarrow AB \rightarrow$ as C. None of the points of r_A^{op} are on that side of AB, so D cannot be on r_A^{op}.





So now we have a quadrilateral $\square ABCD$ with three right angles, $\angle A$, $\angle B$, and $\angle C$. It is actually a convex quadrilateral too (I leave it to you to figure out why), so the diagonal AC divides $\square ABCD$ into two triangles $\triangle ABC$ and $\triangle ADC$. Then, since the angle sum of a triangle is 180°,

$$\begin{gathered} s(\triangle ABC) + s(\triangle ADC) = 180^\circ + 180^\circ \\ (\angle CAB) + (\angle B) + (\angle ACB) + (\angle CAD) + (\angle D) + (\angle ACD) = 360^\circ \\ (\angle A) + (\angle B) + (\angle C) + (\angle D) = 360^\circ \\ 90^\circ + 90^\circ + 90^\circ + (\angle D) = 360^\circ \\ (\angle D) = 90^\circ. \end{gathered}$$

□

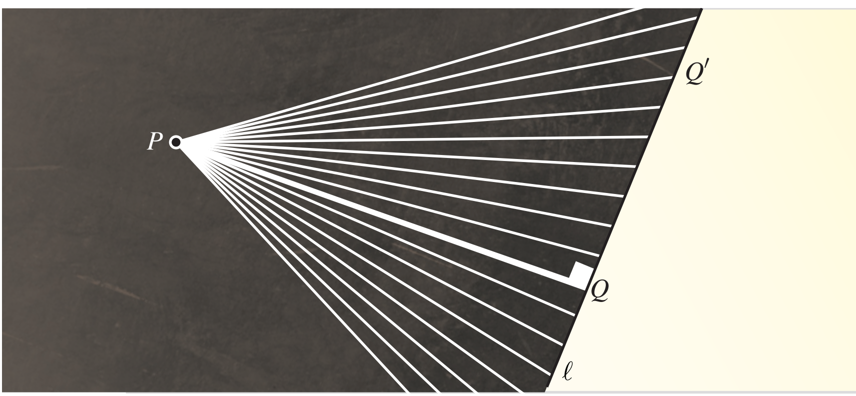
That means that, yes, rectangles *do* exist in Euclidean geometry. In the next lemma, I have listed some basic properties of a rectangle. I will leave it to you to prove these (they aren't hard).

LEM: PROPERTIES OF RECTANGLES
Let $\square ABCD$ be a rectangle. Then
1. $\leftarrow AB \rightarrow$ is parallel to $\leftarrow CD \rightarrow$ and $\leftarrow AD \rightarrow$ is parallel to $\leftarrow BC \rightarrow$
2. $AB \simeq CD$ and $AD \simeq BC$ and $AC \simeq BD$.

For the last result of this section, I would like to get back to parallel lines. One of the things that we will see when we study non-Euclidean geometry is that parallel lines tend to diverge from each other. That doesn't happen in non-Euclidean geometry. It is one of the key differences between the two geometries. Let me make this more precise. Suppose that P is a point which is not on a line ℓ. Define the distance from P to ℓ to be the minimum distance from P to a point on ℓ:

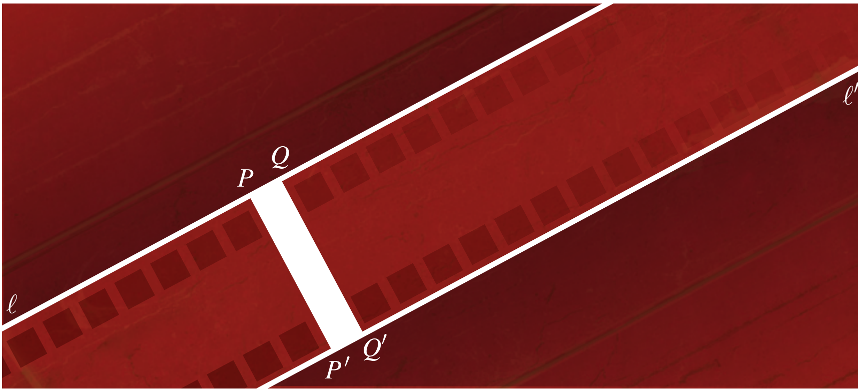
$$d(P,\ell) = \min\left\{ |PQ| \,\middle|\, Q \text{ is on } \ell \right\}.$$

That minimum actually occurs when Q is the foot of the perpendicular to ℓ through P. To see why, let Q' be any other point on ℓ. In $\triangle PQQ'$, the right angle at Q is the largest angle. By the Scalene Triangle Theorem, that means that the opposite side PQ' has to be the longest side, and so $|PQ'| > |PQ|$.



The distance from a point to a line is measured along the segment from the point to the line which is perpendicular to the line.

Now, for a given pair of parallel lines, that distance as measured along perpendiculars does not change.



THM: PARALLEL LINES ARE EVERYWHERE EQUIDISTANT

If ℓ and ℓ' are parallel lines, then the distance from a point on ℓ to ℓ' is constant. In other words, if P and Q are points on ℓ, then

$$d(P,\ell') = d(Q,\ell').$$

Proof. Let P' and Q' be the feet of the perpendiculars on ℓ' from P and Q respectively. That way,

$$d(P,\ell') = |PP'| \quad d(Q,\ell') = |QQ'|.$$

Then $\angle PP'Q'$ and $\angle QQ'P'$ are right angles. By the converse of the Alternate Interior Angle Theorem, $\angle P$ and $\angle Q$ are right angles too– so $\square PQQ'P'$ is a rectangle. Using the previous lemma on rectangles, PP' and QQ', which are the opposite sides of a rectangle, are congruent. $\square$

Exercises

1. Suppose that ℓ_1, ℓ_2 and ℓ_3 are three distinct lines such that: ℓ_1 and ℓ_2 are parallel, and ℓ_2 and ℓ_3 are parallel. Prove then that ℓ_1 and ℓ_3 are parallel.

2. Find the angle sum of a convex n-gon as a function of n.

3. Prove that the opposite sides and the opposite angles of a parallelogram are congruent.

4. Consider a convex quadrilateral $\square ABCD$. Prove that the two diagonals of $\square ABCD$ bisect each other if and only if $\square ABCD$ is a parallelogram.

5. Show that a parallelogram $\square ABCD$ is a rectangle if and only if $AC \simeq BD$.

6. Suppose that the diagonals of a convex quadrilateral $\square ABCD$ intersect one another at a point P and that

   $$AP \simeq BP \simeq CP \simeq DP.$$

   Prove that $\square ABCD$ is a rectangle.

7. Suppose that the diagonals of a convex quadilateral bisect one another at right angles. Prove that the quadrilateral must be a rhombus.

8. Consider a triangle $\triangle ABC$ and three additional points A', B' and C'. Prove that if AA', BB' and CC' are all congruent and parallel to one another then $\triangle ABC \simeq \triangle A'B'C'$.

9. Verify that the Cartesian model (as developed through the exercises in lessons 1 and 3) satisfies Playfair's Axiom.

References

[1] H.S.M. Coxeter and Samuel L. Greitzer. *Geometry Revisited*. Random House, New York, 1st edition, 1967.