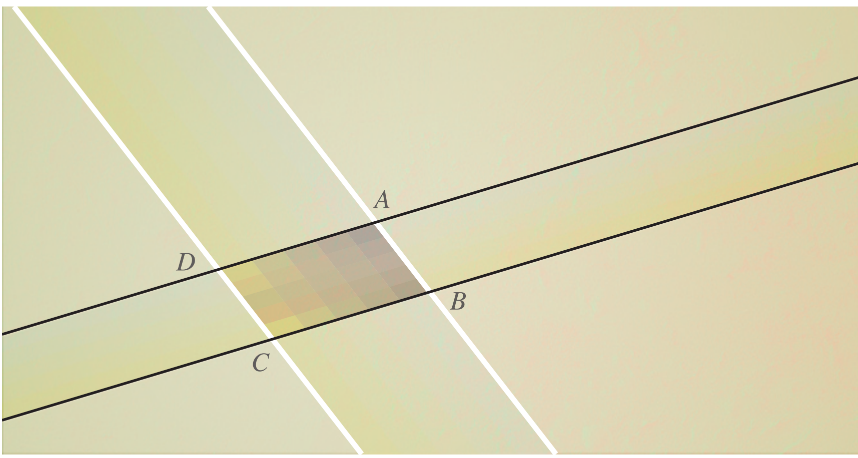
14 PARALLEL PROJECTION

Some calisthenics to start the lesson

In the course of this lesson, we are going to need to use a few facts dealing with parallelograms. First, let me remind of the proper definition of a parallelogram.

DEF: PARALLELOGRAM
A *parallelogram* is a simple quadrilateral whose opposite sides are parallel.



Now on to the facts about parallelograms that we will need for this lesson. None of their proofs are that difficult, but they would be a good warm-up for this lesson.

1 Prove that in a parallelogram, the two pairs of opposite sides are congruent and the two pairs of opposite angles are congruent.

2 Prove that if a convex quadrilateral has one pair of opposite sides which are both parallel and congruent, then it is a parallelogram.

3 Let $\square ABB'A'$ be a simple quadrilateral. Verify that if AA' and BB' are parallel, but AB and $A'B'$ are not, then AA' and BB' cannot be congruent.

Parallel projection

The purpose of this lesson is to introduce a mechanism called parallel projection, a particular kind of mapping from points on one line to points on another. Parallel projection is the piece of machinery that you have to have in place to really understand similarity, which is in turn essential for so much of what we will be doing in the next lessons. The primary goal of this lesson is to understand how distances between points may be distorted by the parallel projection mapping. Once that is figured out, we will be able to turn our attention to the geometry of similarity.

DEF: PARALLEL PROJECTION
A *parallel projection* from one line ℓ to another ℓ' is a map Φ which assigns to each point P on ℓ a point $\Phi(P)$ on ℓ' so that all the lines connecting a point and its image are parallel to one another.

It is easy to construct parallel projections. Any one point P on ℓ and its image $\Phi(P)$ on ℓ' completely determines the projection: for any other point Q on ℓ there is a unique line which passes through Q and is parallel to the line $\leftarrow P\Phi(P) \rightarrow$. Wherever this line intersects ℓ' will have to be $\Phi(Q)$. There are only two scenarios where this construction will not work out: (1) if P is the intersection of ℓ and ℓ', then the lines of projection run parallel to ℓ' and so fail to provide a point of intersection; and (2) if $\Phi(P)$ is the intersection of ℓ and ℓ', then the lines of projection actually coincide rather than being parallel.

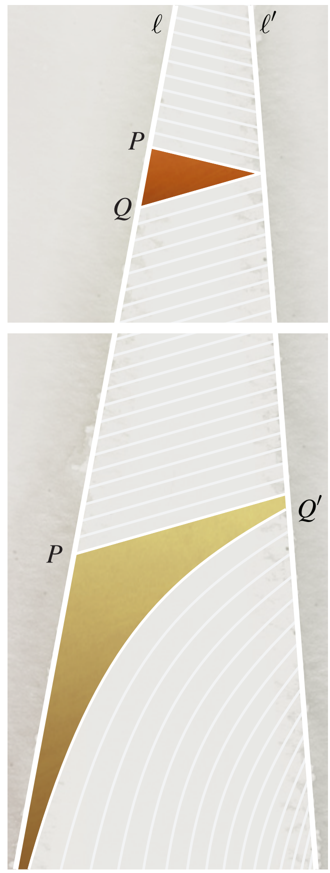
The path from a point P on ℓ to a point P' on ℓ' defines a parallel projection as long as neither P nor P' is the intersection of ℓ and ℓ' (as shown at right).

THM: PARALLEL PROJECTION IS A BIJECTION

A parallel projection is both one-to-one and onto.

Proof. Consider a parallel projection $\Phi : \ell \to \ell'$. First let's see why Φ is one-to-one. Suppose that it is not. That is, suppose that P and Q are two distinct points on ℓ but that $\Phi(P) = \Phi(Q)$. Then the two projecting lines $\leftarrow P\Phi(P) \rightarrow$ and $\leftarrow Q\Phi(Q) \rightarrow$, which ought to be parallel, actually share a point. This can't happen.

Now let's see why Φ is onto, so take a point Q' on ℓ'. We need to make sure that there is a point Q on ℓ so that $\Phi(Q) = Q'$. To get a sense of how Φ is casting points from ℓ to ℓ', let's consider a point P on ℓ and its image $\Phi(P)$ on ℓ'. The projecting line that should lead from Q to Q' ought to be parallel to $\leftarrow P\Phi(P) \rightarrow$. Now, there is a line which passes through Q' and is parallel to $\leftarrow P\Phi(P) \rightarrow$. The only question, then, is whether that line intersects ℓ– if it does, then we have found our Q. What if it doesn't though? In that case, our line is parallel to both $\leftarrow P\Phi(P) \rightarrow$ and ℓ. That would mean that $\leftarrow P\Phi(P) \rightarrow$ and ℓ are themselves parallel. Since P is on both of these lines, we know that cannot be the case. $\square$



Since parallel projection is a bijection, I would like to use a naming convention for the rest of this lesson that I think makes things a little more readable. I will use a prime mark $'$ to indicate the parallel projection of a point. So $\Phi(P) = P'$, $\Phi(Q) = Q'$, and so on.

Parallel projection, order, and congruence.

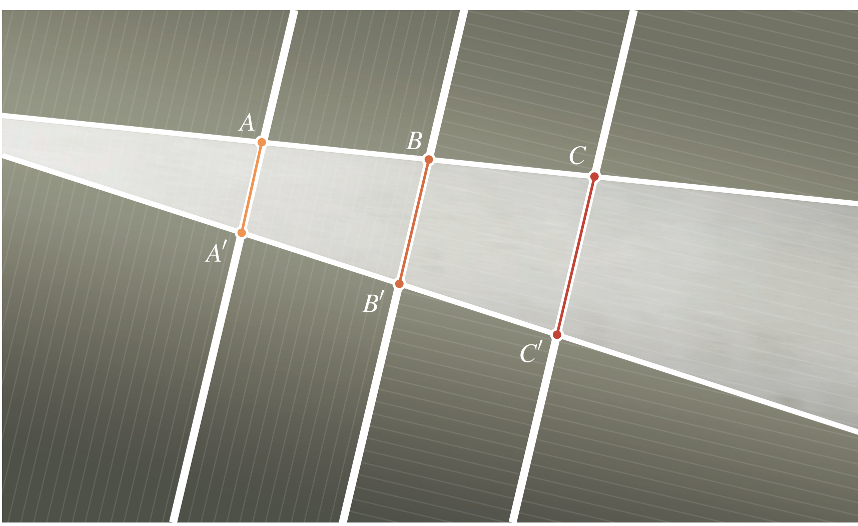
So far we have seen that parallel projection establishes a correspondence between the points of one line and the points of another. What about the order of those points? Can points get shuffled up in the process of a parallel projection? Well, ... no.

THM: PARALLEL PROJECTION AND ORDER
Let $\Phi : \ell \to \ell'$ be a parallel projection. If A, B, and C are points on ℓ and B is between A and C, then B' is between A' and C'.

Proof. Because B is between A and C, A and C must be on opposite sides of the line $\leftarrow BB' \rightarrow$. But:

$\leftarrow AA' \rightarrow$ does not intersect $\leftarrow BB' \rightarrow$ so A' has to be on the same side of $\leftarrow BB' \rightarrow$ as A.

$\leftarrow CC' \rightarrow$ does not intersect $\leftarrow BB' \rightarrow$ so C' has to be on the same side of $\leftarrow BB' \rightarrow$ as C.

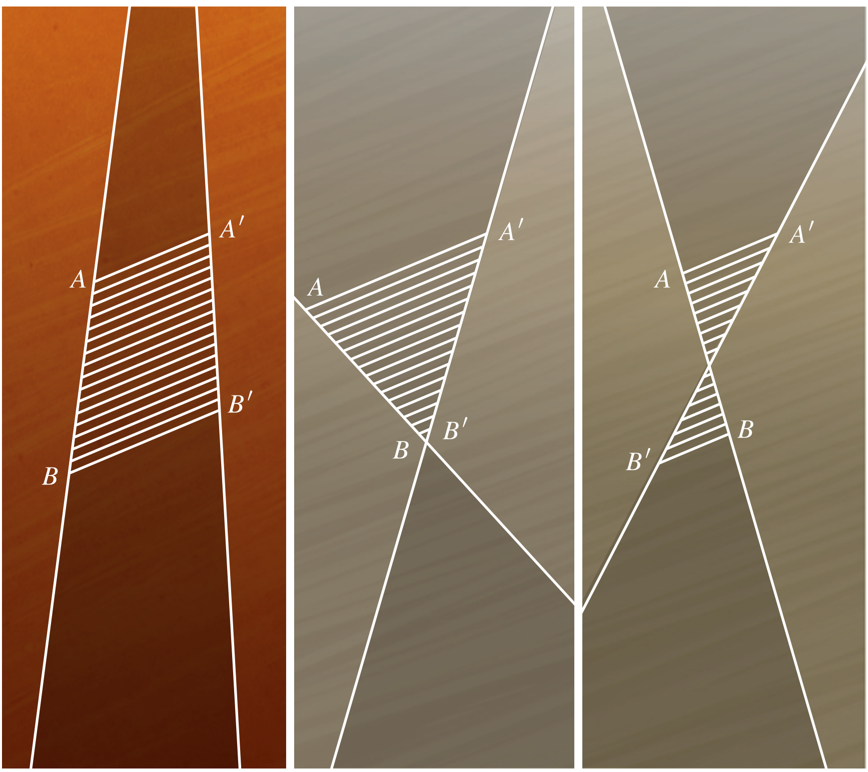


That means A' and C' have to be on opposite sides of $\leftarrow BB' \rightarrow$, and so the intersection of $\leftarrow BB' \rightarrow$ and $A'C'$, which is B', must be between A' and C'. $\square$

That's the story of how parallel projection and order interact. What about congruence?

THM: PARALLEL PROJECTION AND CONGRUENCE
Let $\Phi : \ell \to \ell'$ be a parallel projection. If a, b, A and B are all points on ℓ and if $ab \simeq AB$, then $a'b' \simeq A'B'$.

Proof. There are actually several scenarios here, depending upon the positions of the segments ab and AB relative to ℓ'. They could lie on the same side of ℓ', or they could lie on opposite sides of ℓ', or one or both could straddle ℓ', or one or both could have an endpoint on ℓ'. You have



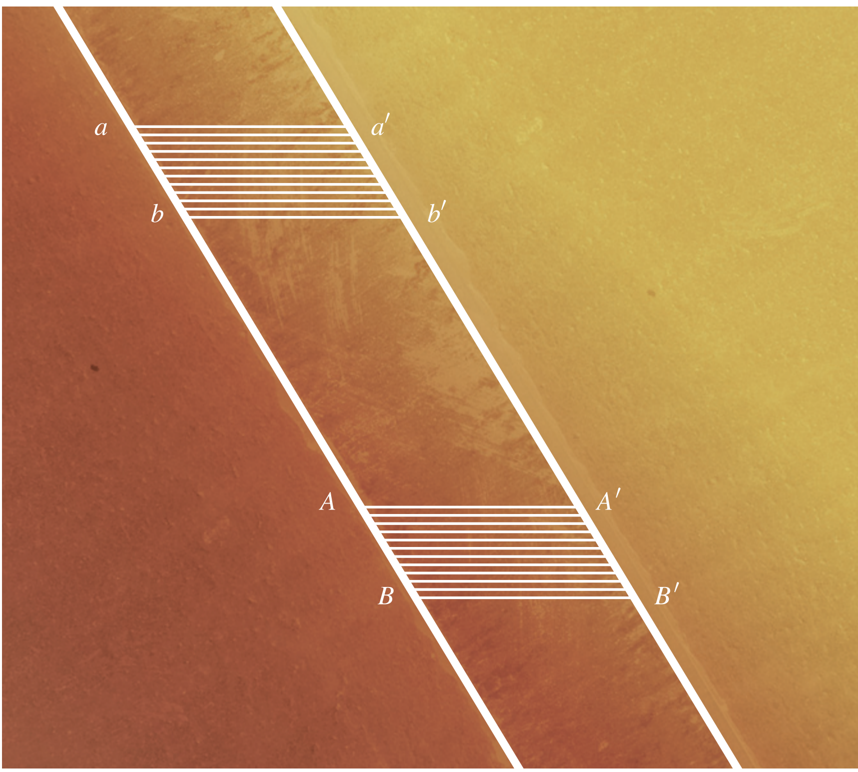
There are three positions for A and B relative to the image line– both on the same side, one on the image line, or one on each side. Likewise, there are three positions for a and b. Therefore, in all, there are nine scenarios.

to handle each of those scenarios slightly differently, but I am only going to address what I feel is the most iconic situation– the one where both segments are on the same side of ℓ'.

Case 1: ℓ and ℓ' are parallel.
First let's warm up with a simple case which I think helps illuminate the more general case– it is the case where ℓ and ℓ' are themselves parallel. Notice all the parallel line segments:

aa' is parallel to bb' and ab is parallel to $a'b'$ so $\square aa'b'b$ is a parallelogram;

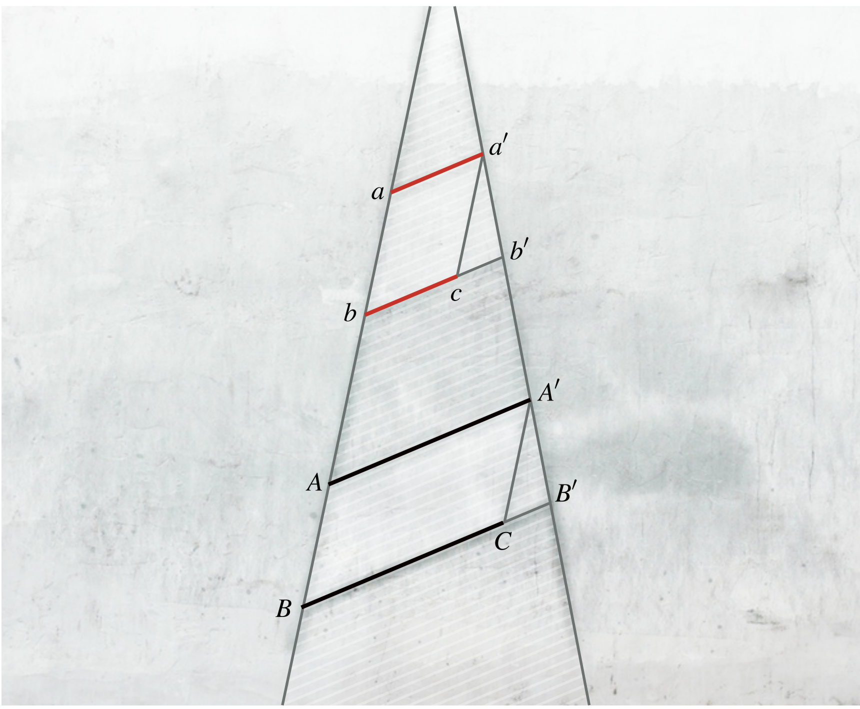
AA' is parallel to BB' and AB is parallel to $A'B'$ so $\square AA'B'B$ is also a parallelogram.



Case 1: when the two lines are parallel.

Because the opposite sides of a parallelogram are congruent (exercise 1 at the start of the lesson), $a'b' \simeq ab$ and $AB \simeq A'B'$. Since $ab \simeq AB$, that means $a'b' \simeq A'B'$.

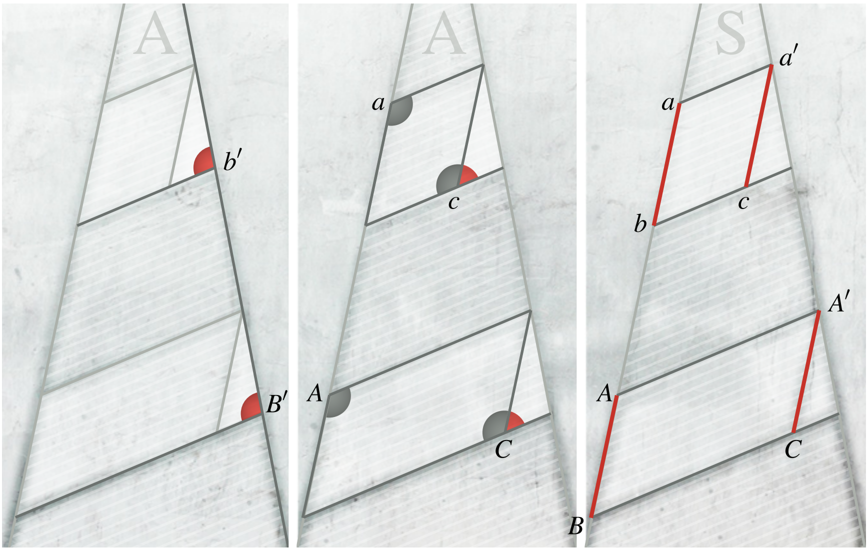
Case 2: ℓ and ℓ' are not parallel.
This is the far more likely scenario. In this case the two quadrilaterals $\square aa'b'b$ and $\square AA'B'B$ will not be parallelograms. I want to use the same approach here as in Case 1 though, so to do that we will need to build some parallelograms into the problem. Because ℓ and ℓ' are not parallel, the segments aa' and bb' cannot be the same length (exercise 3 at the start of this lesson), and the segments AA' and BB' cannot be the same length. Let's assume that aa' is shorter than bb' and that AA' is shorter than BB'. If this is not the case, then it is just a matter of switching some labels to make it so.



Then

- ∘ there is a point c between b and b' so that $bc \simeq aa'$, and
- ∘ there is a point C between B and B' so that $BC \simeq AA'$.

This creates four shapes of interest– the two quadrilaterals $\square a'abc$ and $\square A'ABC$ which are actually parallelograms (exercise 2), and the two triangles $\triangle a'b'c$ and $\triangle A'B'C$. The key here is to prove that $\triangle a'b'c \simeq \triangle A'B'C$. I want to use A·A·S to do that.



[A] $\angle b' \simeq \angle B'$.
The lines cb' and CB' are parallel (they are two of the projecting lines) and they are crossed by the tranversal ℓ'. By the converse of the Alternate Interior Angle Theorem, that means $\angle a'b'c$ and $\angle A'B'C$ are congruent.

[A] $\angle c \simeq \angle C$.
The opposite angles of the two parallelograms are congruent. Therefore $\angle a'cb \simeq \angle a'ab$ and $\angle A'AB \simeq \angle A'CB$. But aa' and AA' are parallel lines cut by the transversal ℓ, so $\angle a'ab \simeq \angle A'AB$. That means that $\angle a'cb \simeq \angle A'CB$, and so their supplements $\angle a'cb'$ and $\angle A'CB'$ are also congruent.

[S] $a'c \simeq A'C$.
The opposite sides of the two parallelograms are congruent too. Therefore $a'c \simeq ab$ and $AB \simeq A'C$, and since $ab \simeq AB$, that means $a'c \simeq A'C$.

By A·A·S, then, $\triangle a'b'c \simeq \triangle A'B'C$. The corresponding sides $a'b'$ and $A'B'$ have to be congruent.

□

Parallel projection and distance

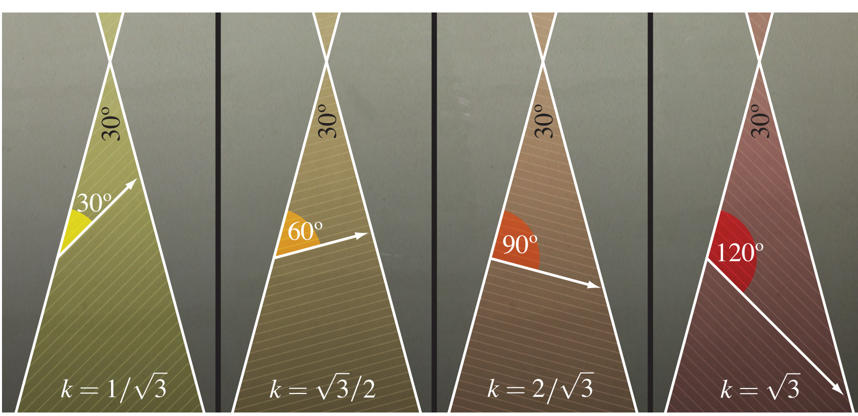
That brings us to the question at the very heart of parallel projection. If Φ is a parallel projection and A and B are two points on ℓ, how do the lengths $|AB|$ and $|A'B'|$ compare? In Case 1 of the last proof, the segments AB and $A'B'$ ended up being congruent, but that was because ℓ and ℓ' were parallel. In general, AB and $A'B'$ do not have to be congruent. But (and this is the key) in the process of parallel projecting from one line to another, all distances are scaled by a constant multiple.

THM: PARALLEL PROJECTION AND DISTANCE
If $\Phi : \ell \to \ell'$ is a parallel projection, then there is a constant k such that

$$|A'B'| = k|AB|$$

for all points A and B on ℓ.

I want to talk about a few things before diving in after the formal proof. The first is that the previous theorem on congruence gives us a way to narrow the scope of the problem. Fix a point O on ℓ and let r be one of the two rays along ℓ with O as its endpoint. The Segment Construction Axiom says that every segment AB on ℓ is congruent to a segment OP where P is some point on r. We have just seen that parallel projection maps congruent segments to congruent segments. So if Φ scales all segments of the form OP by a factor of k, then it must scale all the segments of ℓ by that same factor.



Some parallel projections and their scaling constants.

The second deals with parallel projecting end-to-end congruent copies of a segment. For this, let me introduce another convenient notation convention: for the rest of this argument, when I write a point with a subscript P_d, the subscript d is the distance from that point to O. Now, pick a particular positive real value x, and let

$$k = |O'P'_x|/|OP_x|,$$

so that Φ scales the segment OP_x by a factor of k. Of course, eventually we will have to show that Φ scales all segments by that same factor, but for now let's restrict our attention to the segments OP_{nx}, where n is a positive integer. Between O and P_{nx} are $P_x, P_{2x}, \ldots P_{(n-1)x}$ in order:

$$O * P_x * P_{2x} * \cdots * P_{(n-1)x} * P_{nx}.$$

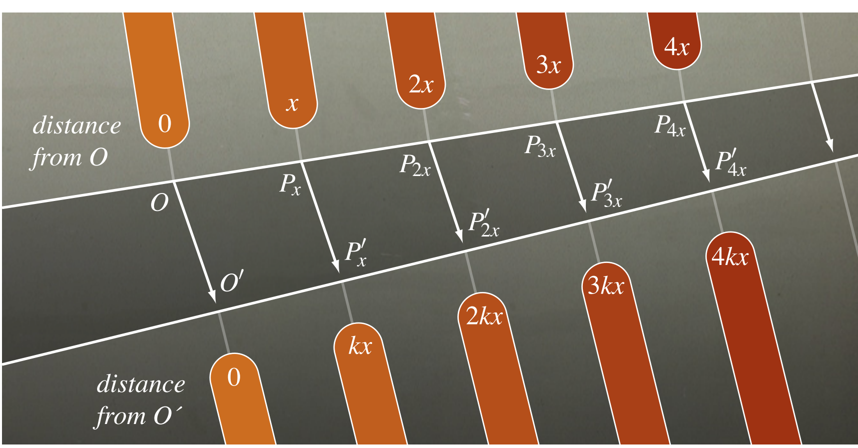
We have seen that parallel projection preserves the order of points, so

$$O' * P'_x * P'_{2x} * \cdots * P'_{(n-1)x} * P'_{nx}.$$

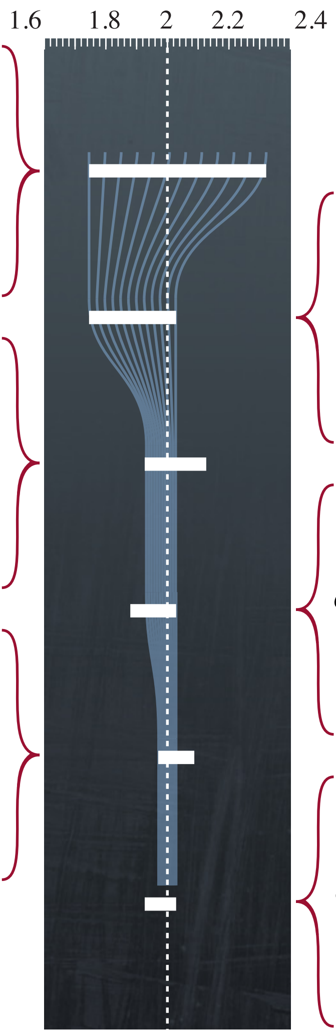
Each segment $P_{ix}P_{(i+1)x}$ is congruent to OP_x and consequently each parallel projection $P'_{ix}P'_{(i+1)x}$ is congruent to $O'P'_x$. Just add them all together

$$\begin{aligned} |O'P'_{nx}| &= |O'P'_x| + |P'_xP'_{2x}| + |P'_{2x}P'_{3x}| + \cdots + |P'_{(n-1)x}P'_{nx}| \\ &= kx + kx + kx + \cdots + kx \quad (n \text{ times}) \\ &= k \cdot nx \end{aligned}$$

and so Φ scales OP_{nx} by a factor of k.



Sadly, no matter what x is, the points P_{nx} account for an essentially inconsequential portion of the set of all points of r. However, if OP_x and OP_y were to have two different scaling factors we could use this end-to-end copying to magnify the difference between them. The third thing I would like to do, then, is to look at an example to see how this actually works, and how this ultmately prevents there from being two different scaling factors. In this example, let's suppose that $|O'P_1'| = 2$, so that all integer length segments on ℓ are scaled by a factor of 2, and let's take a look at what this means for $P_{3.45}$. Let k be the scaling factor for $OP_{3.45}$ and let's see what the first few end-to-end copies of $OP_{3.45}$ tell us about k.



1

$3 < 3.45 < 4$

$O * P_3 * P_{3.45} * P_4$

$O' * P_3' * P_{3.45}' * P_4'$

$6 < 3.45k < 8$

$1.74 < k < 2.32$

2

$6 < 6.9 < 7$

$O * P_6 * P_{6.9} * P_7$

$O' * P_6' * P_{6.9}' * P_7'$

$12 < 6.9k < 14$

$1.74 < k < 2.03$

3

$10 < 10.35 < 11$

$O * P_{10} * P_{10.35} * P_{11}$

$O' * P_{10}' * P_{10.35}' * P_{11}'$

$20 < 10.35k < 22$

$1.93 < k < 2.13$

4

$13 < 13.8 < 14$

$O * P_{13} * P_{13.8} * P_{14}$

$O' * P_{13}' * P_{13.8}' * P_{14}'$

$26 < 13.8k < 28$

$1.88 < k < 2.03$

5

$17 < 17.25 < 18$

$O * P_{17} * P_{17.25} * P_{18}$

$O' * P_{17}' * P_{17.25}' * P_{18}'$

$34 < 17.25k < 36$

$1.97 < k < 2.09$

6

$20 < 20.7 < 21$

$O * P_{20} * P_{20.7} * P_{21}$

$O' * P_{20}' * P_{20.7}' * P_{21}'$

$40 < 20.7k < 42$

$1.93 < k < 2.03$

notation

The floor function, $f(x) = \lfloor x \rfloor$, assigns to each real number x the largest integer which is less than or equal to it.

The ceiling function, $f(x) = \lceil x \rceil$, assigns to each real number x the smallest integer which is greater than or equal to it.

Proof. It is finally time to prove that parallel projection scales distance. Let $k = |O'P_1'|$ so that k is the scaling factor for the segment of length one (and consequently all integer length segments). Now take some arbitrary point P_x on ℓ and let k' be the scaling factor for the segment OP_x. We want to show that $k' = k$ and to do that, I want to follow the same basic strategy as in the example above– capture k' in an increasingly narrow band around k by looking at the parallel projection of P_{nx} as n increases.

$$\lfloor nx \rfloor < nx < \lceil nx \rceil$$

$$O * P_{\lfloor nx \rfloor} * P_{nx} * P_{\lceil nx \rceil}$$

$$O' * P'_{\lfloor nx \rfloor} * P'_{nx} * P'_{\lceil nx \rceil}$$

$$k\lfloor nx \rfloor < k'nx < k\lceil nx \rceil$$

$$k(nx-1) < k\lfloor nx \rfloor < k'nx < k\lceil nx \rceil < k(nx+1) \qquad *$$

$$k(nx-1) < k'nx < k(nx+1)$$

$$k \cdot (nx-1)/(nx) < k' < k \cdot (nx+1)/(nx)$$

As n increases, the two ratios $(nx-1)/(nx)$ and $(nx+1)/(nx)$ both approach 1. In the limit as n goes to infinity, they are one. Since the above inequalities have to be true for all n, the only possible value for k' , then, is k. □

* *In this step, I have replaced one set of inequalities with another, less precise, set. The new inequalities are easier to manipulate mathematically though, and are still accurate enough to get the desired result.*

Exercises

1. Investigate the other possible cases in the proof that parallel projection preserves order.

2. Suppose that Φ is a parallel projection from ℓ to ℓ'. If ℓ and ℓ' intersect, and that point of intersection is P, prove that $\Phi(P) = P$.

3. Prove that if ℓ and ℓ' are parallel, then the scaling factor of any parallel projection between them must be one, but that if ℓ and ℓ' are not parallel, then there is a parallel projection with every possible scaling factor k where $0 < k < \infty$.

4. In the lesson 7, we constructed a distance function, and one of the keys to that construction was locating the points on a ray which were a distance of $m/2^n$ from its endpoint. In Euclidean geometry, there is a construction which locates all the points on a ray which are *any* rational distance m/n from its endpoint. Take two (non-opposite) rays r and r' with a common endpoint O. Along r, lay out m congruent copies of a segment of length one, ending at the point P_m. Along r', lay out n congruent copies of a segment of length one, ending at the point Q_n. Mark the point Q_1 on r' which is a distance one from O. Verify that the line which passes through Q_1 and is parallel to P_mQ_n intersects r a distance of m/n from O.