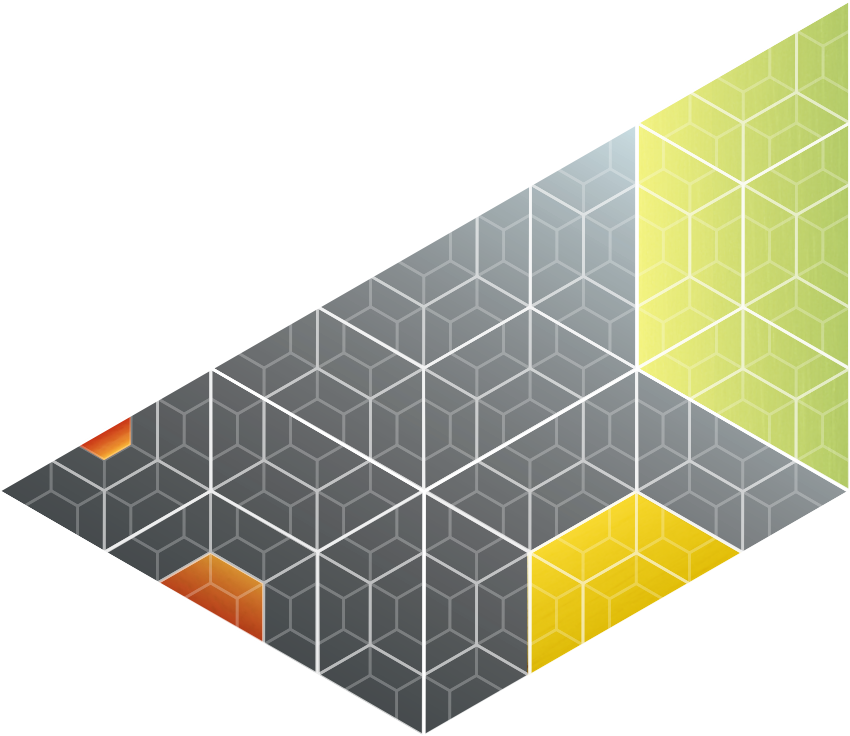# 15 SIZE IS RELATIVE   **SIMILARITY**

In the lessons on neutral geometry, we spent a lot of effort to gain an understanding of polygon congruence. In particular, I think we were pretty thorough in our investigation of triangle and quadrilateral congruence. So I sincerely hope that you haven't forgotten what it means for two polygons to be congruent:

     1. all their corresponding sides must be congruent, and
     2. all their corresponding interior angles must be congruent.

Remember as well that polygon congruence is an equivalence relation (it is reflexive, symmetric, and transitive). It turns out that congruence is not the only important equivalence relation between polygons, though, and the purpose of this lesson is to investigate another: similarity.

    Similarity is a less demanding relation than congruence. I think of congruent polygons as exactly the same, just positioned differently. I think of similar polygons as "scaled versions" of one another– the same shape, but possibly different sizes. That's not really a definition, though, so let's get to something a little more formal.
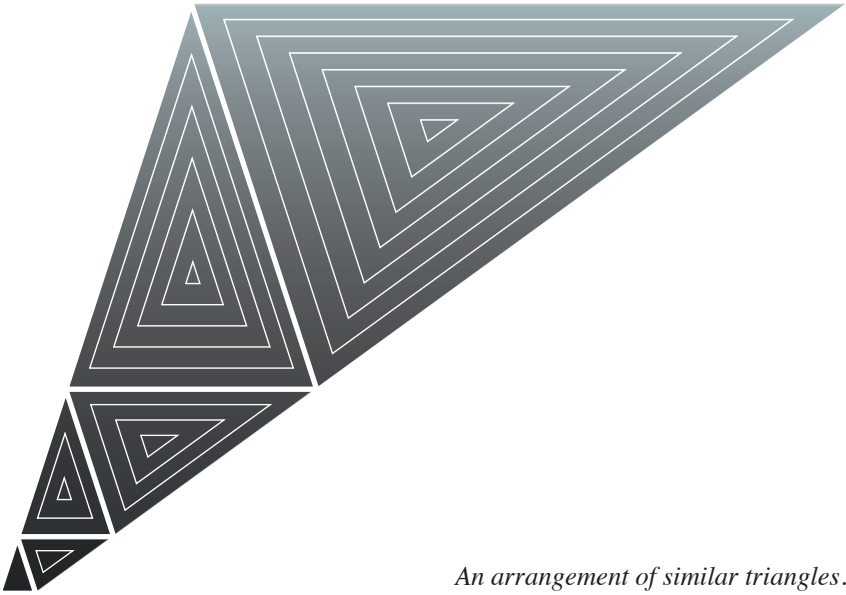
DEF: SIMILAR POLYGONS
Two n-sided polygons $P_1P_2\ldots P_n$ and $Q_1Q_2\ldots Q_n$ are *similar* to one another if they meet two sets of conditions

  1. corresponding interior angles are congruent:

$$\angle P_i \simeq \angle Q_i, \quad 1 \le i \le n.$$

  2. corresponding side lengths differ by the same constant multiple:

$$|P_iP_{i+1}| = k \cdot |Q_iQ_{i+1}|, \quad 1 \le i \le n.$$

An arrangement of similar triangles.

I will use the notation $P_1 P_2 \ldots P_n \sim Q_1 Q_2 \ldots Q_n$ to indicate similarity. There are a few things worth noting here. First, if polygons are congruent, they will be similar as well– the scaling constant k will be one in this case. Second, similarity *is* an equivalence relation– I leave it to you to verify that the three required conditions are met. Third, when you jump from one polygon to another similar polygon, all the corresponding segments lengths are scaled by the same amount. That behavior echoes the work we did in the last lesson, and for good reason: parallel projection underlies everything that we are going to do in this lesson.

A spiralling stack of similar golden rectangles (see the exercises).

Much of the time, when working with either parallel projection or simi-
larity, the actual scaling constant is just not that important. The only thing
that matters is that there *is* a scaling constant. Fortunately, the existence
of a scaling constant can be indicated without ever mentioning what it is.
The key to doing this is ratios. Consider a parallel projection from line ℓ
to line ℓ'. Let A, B, a, and b be points on ℓ and let A', B', a' and b' be their
respective images on ℓ'. The main result of the last lesson was that there
is a scaling constant k so that

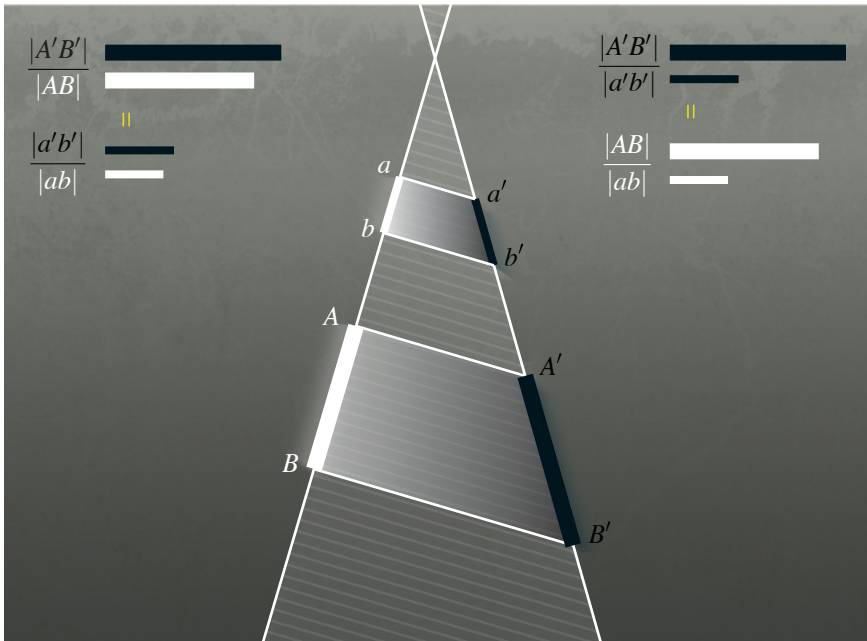$$|A'B'| = k \cdot |AB| \quad \& \quad |a'b'| = k \cdot |ab|.$$

The ratios I am talking about are only a step away from this pair of equa-
tions.

Ratio 1: Solve for k in both equa-
tions and set them equal to each
other

$$\frac{|A'B'|}{|AB|} = \frac{|a'b'|}{|ab|}.$$

Ratio 2: Starting from the first ra-
tio, multiply through by $|AB|$ and
divide through by $|a'b'|$

$$\frac{|A'B'|}{|a'b'|} = \frac{|AB|}{|ab|}.$$

Two invariant ratios of a parallel projection.
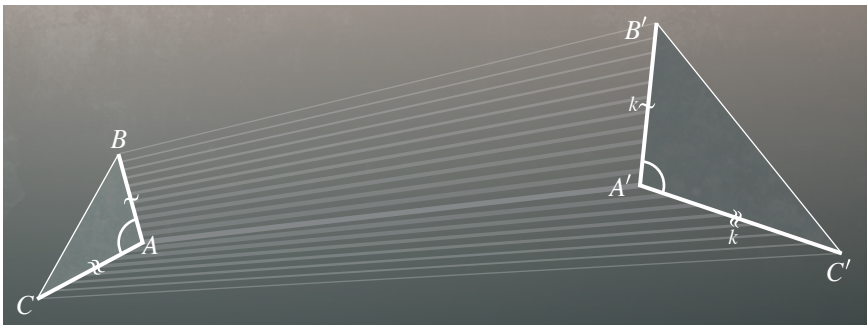
Triangle similarity theorems

I would now like to turn our attention to a few theorems that deal with similarity of triangles. I like to think of these similarity theorems as degenerations of the triangle congruence theorems, where the strict condition of side congruence, $A'B' \simeq AB$, is replaced with the more flexible condition of constant scaling, $|A'B'| = k|AB|$. First up is the S·A·S similarity theorem.

THM: S·A·S SIMILARITY
In triangles $\triangle ABC$ and $\triangle A'B'C'$, if $\angle A \simeq \angle A'$ and if there is a constant k so that

$$|A'B'| = k \cdot |AB| \quad \& \quad |A'C'| = k \cdot |AC|,$$

then $\triangle ABC \sim \triangle A'B'C'$.

Proof. First of all, let me point out that just as with the parallel projection, the second condition in the S·A·S similarity theorem can be recast in terms of ratios:

$$\begin{cases} |A'B'| = k|AB| \\ |A'C'| = k|AC| \end{cases} \quad \Longleftrightarrow \quad \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} \quad \Longleftrightarrow \quad \frac{|A'B'|}{|A'C'|} = \frac{|AB|}{|AC|}.$$

With that said, what we need to do in this proof is to establish two more angle congruences, that $\angle B \simeq \angle B'$ and $\angle C \simeq \angle C'$, and one more ratio of sides, that $|B'C'| = k|BC|$. Two parallel projections will form the backbone of this proof. The first will establish the two angle congrunces while the second will calculate the ratio of the third pair of sides.
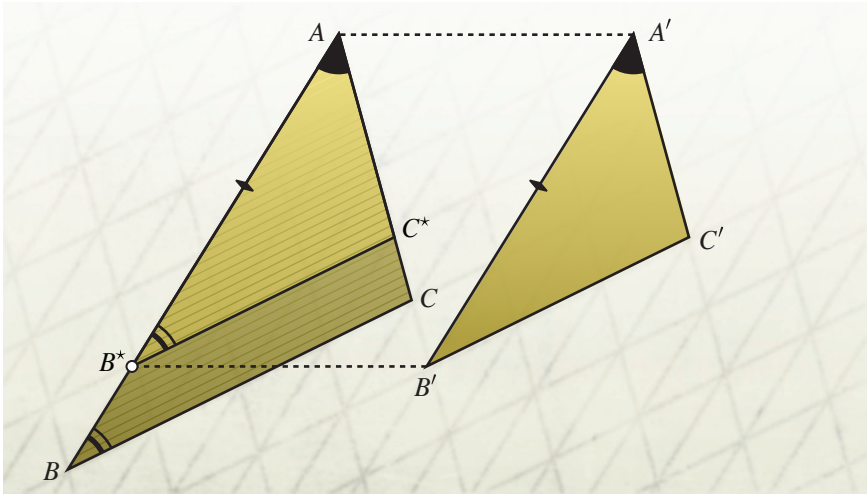
The first parallel projection. The primary purpose of the first projection is to build a transitional triangle which is congruent to $\triangle A'B'C'$ but positioned on top of $\triangle ABC$. Begin by locating the point $B^\star$ on $AB \rightarrow$ so that $AB^\star \simeq A'B'$. We cannot know the exact location of $B^\star$ relative to B on this ray– that depends upon whether $A'B'$ is shorter or longer than AB. For this argument, assume that $A'B'$ is shorter than AB, which will place $B^\star$ between A and B (the other case is not substantially different). Consider the parallel projection

$$\Phi_1 : (\leftarrow AB \rightarrow) \longrightarrow (\leftarrow AC \rightarrow)$$

which takes B to C. Note that since A is the intersection of these two lines, $\Phi_1(A) = A$. Label $C^\star = \Phi_1(B^\star)$. Let's see how the newly formed $\triangle AB^\star C^\star$ compares with $\triangle A'B'C'$. Compare the ratios

$$\frac{|AC^\star|}{|AC|} \overset{1}{=} \frac{|AB^\star|}{|AB|} \overset{2}{=} \frac{|A'B'|}{|AB|} \overset{3}{=} \frac{|A'C'|}{|AC|}.$$

1. parallel projection
2. constructed congruence
3. given

If you look at the first and last entries in that string of equalities you will see that $|AC^\star| = |A'C'|$. Put that together with what we already knew, that $AB^\star \simeq A'B'$ and $\angle A \simeq \angle A'$, and by S·A·S, we see that $\triangle A'B'C'$ and $\triangle AB^\star C^\star$ are congruent. In particular, that means $\angle B' \simeq \angle B^\star$ and $\angle C' \simeq$

$\angle C^\star$. Now let's turn back to see how $\triangle AB^\star C^\star$ relates to $\triangle ABC$. In order to locate $C^\star$, we used a projection which was parallel to $\leftarrow BC \rightarrow$. That of course means $\leftarrow B^\star C^\star \rightarrow$ and $\leftarrow BC \rightarrow$ are parallel to one another, and so, by the converse of the Alternate Interior Angle Theorem, $\angle B^\star \simeq \angle B$ and $\angle C^\star \simeq \angle C$. Since angle congruence is transitive, we now have the two required angle congruences, $\angle B \simeq \angle B'$ and $\angle C \simeq \angle C'$.
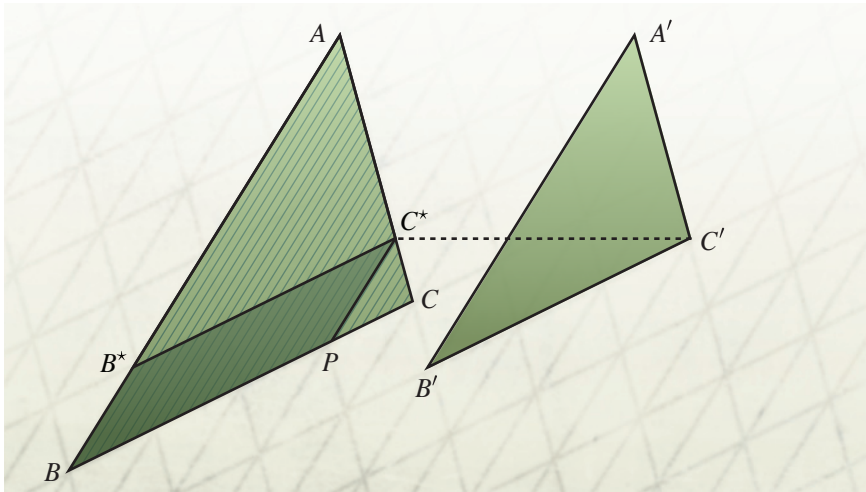
The second parallel projection. Consider the parallel projection

$$\Phi_2 : (\leftarrow AC \rightarrow) \longrightarrow (\leftarrow BC \rightarrow)$$

which maps A to B. Again, since C is the intersection of those two lines, $\Phi_2(C) = C$. The other point of interest this time is $C^\star$. Define $P = \Phi_2(C^\star)$. In doing so, we have effectively carved out a parallelogram $BB^\star C^\star P$. Recall that the opposite sides of a parallelogram are congruent– in particular, $B^\star C^\star \simeq BP$. Now consider the ratios that Φ_2 provides

$$\frac{|B'C'|}{|BC|} \overset{1}{=} \frac{|B^\star C^\star|}{|BC|} \overset{2}{=} \frac{|BP|}{|BC|} \overset{3}{=} \frac{|AC^\star|}{|AC|} \overset{4}{=} \frac{|A'C'|}{|AC|} = k.$$

1. triangle congruence established above
2. opposite sides of a parallelogram
3. parallel projection
4. triangle congruence established above

Thus, $|B'C'| = k|BC|$, as needed. $\qquad\qquad\qquad\qquad\qquad\square$

Back in the neutral geometry lessons, after S·A·S we next encountered A·S·A and A·A·S. Unlike S·A·S, both of those theorems reference only one pair of sides in the triangles. Let's take a look at what happens when you try to modify those congruence conditions into similarity conditions.

A·S·A Congruence	A·S·A Similarity (?)
$\angle A \simeq \angle A'$	$\angle A \simeq \angle A'$
$AB \simeq A'B'$	$|A'B'| = k \cdot |AB|$
$\angle B \simeq \angle B'$	$\angle B \simeq \angle B'$

A·A·S Congruence	A·A·S Similarity (?)
$\angle A \simeq \angle A'$	$\angle A \simeq \angle A'$
$\angle B \simeq \angle B'$	$\angle B \simeq \angle B'$
$BC \simeq B'C'$	$|B'C'| = k \cdot |BC|$

In each of these conversions, the condition on the one side is automatically satisfied– there will always be a real value of k that makes the equation true. That is a hint that it may take only two angle congruences to guarantee similarity.
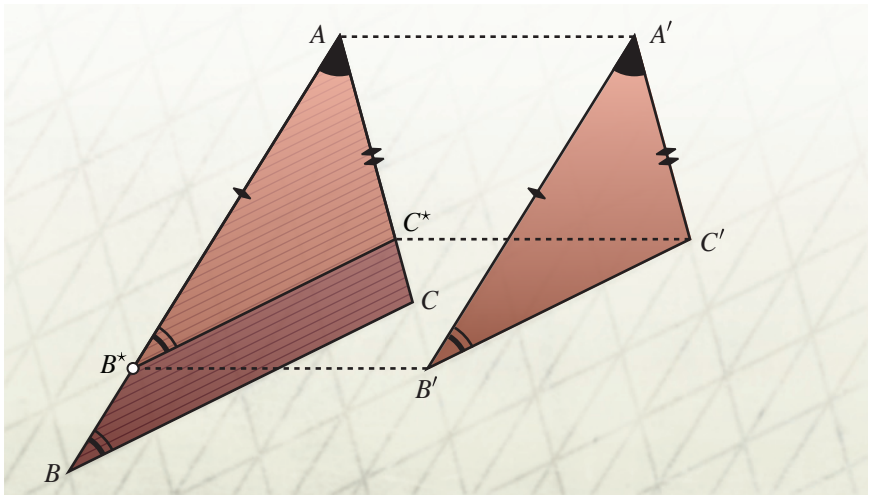
THM: A · A SIMILARITY
In triangles $\triangle ABC$ and $\triangle A'B'C'$, if $\angle A \simeq \angle A'$ and $\angle B \simeq \angle B'$, then $\triangle ABC \sim \triangle A'B'C'$.
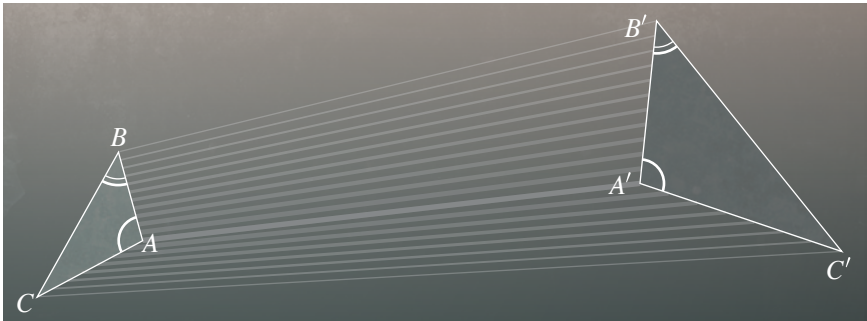
Proof. We have plenty of information about the angles, so what we need here is some information about ratios of sides. In particular, I want to show that

$$\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|}.$$

Along with the given congruence $\angle A \simeq \angle A'$, that will be enough to use S·A·S similarity. As in the S·A·S similarity proof, I want to construct a transition triangle: one that is positioned on top of $\triangle ABC$ but is congruent to $\triangle A'B'C'$. To do that, locate $B^\star$ on $AB \rightarrow$ so that $AB^\star \simeq A'B'$, and $C^\star$ on $AC \rightarrow$ so that $AC^\star \simeq A'C'$. By S·A·S, $\triangle AB^\star C^\star$ and $\triangle A'B'C'$ are congruent. Now take a look at all the congruent angles

$$\angle B^\star \simeq \angle B' \simeq \angle B.$$

According to the Alternate Interior Angle Theorem, $\leftarrow B^\star C^\star \rightarrow$ and $\leftarrow BC \rightarrow$ must be parallel. Therefore the parallel projection from $\leftarrow AB \rightarrow$ to $\leftarrow AC \rightarrow$ which maps B to C and A to itself will also map $B^\star$ to $C^\star$. That gives us some ratios

$$\frac{|A'B'|}{|AB|} \overset{1}{=} \frac{|AB^\star|}{|AB|} \overset{2}{=} \frac{|AC^\star|}{|AC|} \overset{3}{=} \frac{|A'C'|}{|AC|}.$$

1. constructed congruence
2. parallel projection
3. constructed congruence

The first and last terms in that list of equalities give the ratio we need. That, together with the known congruence $\angle A \simeq \angle A'$, is enough for S·A·S similarity, so $\triangle ABC \sim \triangle A'B'C'$. $\qquad \square$

Note that while A·A·A was not enough to guarantee congruence, thanks to the result above, we now know that it is (more than) enough to guarantee similarity. Finally, the last of the triangle similarity theorems is S·S·S similarity (S·S·A, which just misses as a congruence theorem, is done in again by the same counterexample).

THM: S·S·S SIMILARITY
In triangles $\triangle ABC$ and $\triangle A'B'C'$, if there is a constant k so that

$$|A'B'| = k \cdot |AB| \quad |B'C'| = k \cdot |BC| \quad \& \quad |C'A'| = k \cdot |CA|,$$

then $\triangle ABC \sim \triangle A'B'C'$.
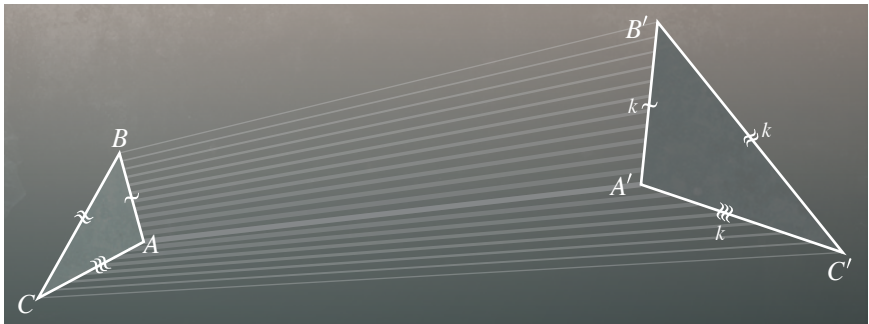
I am going to leave the proof of this last similarity theorem as an exercise for you.
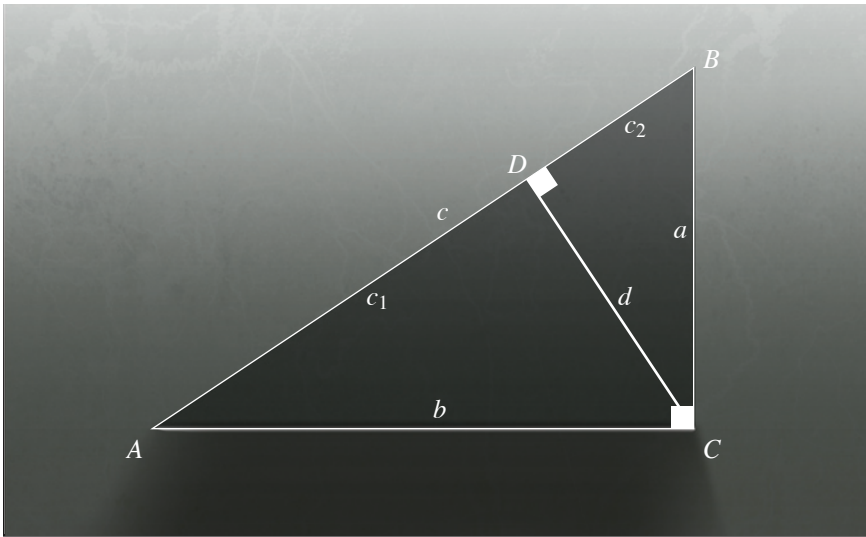
The Pythagorean Theorem

Before we close this lesson, though, let's meet one of the real celebrities of the subject.

THM: THE PYTHAGOREAN THEOREM
Let $\triangle ABC$ be a right triangle whose right angle is at the vertex C. Identify the lengths of each side as

$$a = |BC| \quad b = |AC| \quad c = |AB|.$$

Then $c^2 = a^2 + b^2$.

A proof of the Pythagorean Theorem via similarity.

Proof. There are many, many proofs of this theorem. The one that I am going to give involves dividing the triangle into two smaller triangles, showing each of those is similar to the initial triangle, and then working with ratios. Let D be the foot of the perpendicular to AB through C. The segment CD divides $\triangle ABC$ into two smaller triangles: $\triangle ACD$ and $\triangle BCD$. Let's go ahead and label the lengths of the newly created sides of those two triangles:

$$c_1 = |AD| \quad c_2 = |BD| \quad d = |CD|$$

and note that $c = c_1 + c_2$. Now $\triangle ADC$ shares $\angle A$ with $\triangle ACB$, and they both have a right angle, so by the A·A similarity theorem, $\triangle ADC \sim \triangle ACB$. Similarly, $\triangle BDC$ shares $\angle B$ with $\triangle ACB$, and they both have a right angle as well, so again by A·A similarity, $\triangle BDC \sim \triangle ACB$. From these similarities, there are many ratios, but the two that we need are

$$\frac{a}{c} = \frac{c_2}{a} \implies a^2 = c \cdot c_2 \quad \& \quad \frac{b}{c} = \frac{c_1}{b} \implies b^2 = c \cdot c_1.$$

Now all you have to do is add those two equations together and simplify to get the Pythagorean Theorem

$$a^2 + b^2 = c \cdot c_2 + c \cdot c_1 = c(c_2 + c_1) = c^2.$$

□

Exercises

1. Prove that similarity of polygons is an equivalence relation.

2. Prove the S·S·S triangle similarity theorem.

3. State and prove the S·A·S·A·S and A·S·A·S·A similarity theorems for convex quadrilaterals.

The six trigonometric functions can be defined, for values of θ between 0 and $90°$, as ratios of pairs of sides of a right triangle with an interior angle θ. If the length of the hypotenuse is h, the length of the leg adjacent to θ is a, and the length of the leg opposite θ is o, then these functions are defined as

$$\text{sine: } \sin(\theta) = o/h$$
$$\text{cosine: } \cos(\theta) = a/h$$
$$\text{tangent: } \tan(\theta) = o/a$$
$$\text{cotangent: } \cot(\theta) = a/o$$
$$\text{secant: } \sec(\theta) = h/a$$
$$\text{cosecant: } \csc(\theta) = h/o.$$

4. Verify that the six trigonometric functions are well-defined. That is, show that it does not matter which right triangle with interior angle θ you choose– these six ratios will not change.

5. Verify the Pythagorean identities (for values of θ between 0 and $90°$).

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

6. Verify the cofunction identities (for values of θ between 0 and 90°).

$$\sin(90° - \theta) = \cos\theta$$
$$\cos(90° - \theta) = \sin\theta$$
$$\tan(90° - \theta) = \cot\theta$$
$$\cot(90° - \theta) = \tan\theta$$
$$\sec(90° - \theta) = \csc\theta$$
$$\csc(90° - \theta) = \sec\theta$$

7. The geometric mean of two numbers a and b is defined to be $\sqrt{ab}$. Let $\triangle ABC$ be a right triangle with right angle at C and let D be the point on AB so that CD is perpendicular to AB (the same setup as in the proof of the Pythagorean Theorem). Verify that $|CD|$ is the geometric mean of $|CA|$ and $|CB|$.

8. Consider a rectangle $\square ABCD$ with $|AB| < |BC|$, and suppose that this rectangle has the following special property: if a square $\square ABEF$ is constructed inside $\square ABCD$, then the remaining rectangle $\square ECDF$ is similar to the original $\square ABCD$. A rectangle with this property is called a *golden rectangle*. Find the value of $|BC|/|AB|$, a value known as the golden ratio.