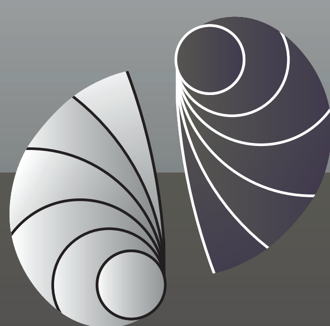
...COMES AROUND



17 CIRCUMFERENCE.

A theorem on perimeters

In the lesson on polygons, I defined the perimeter of a polygon $\mathcal{P} = P_1 \cdots P_n$ as

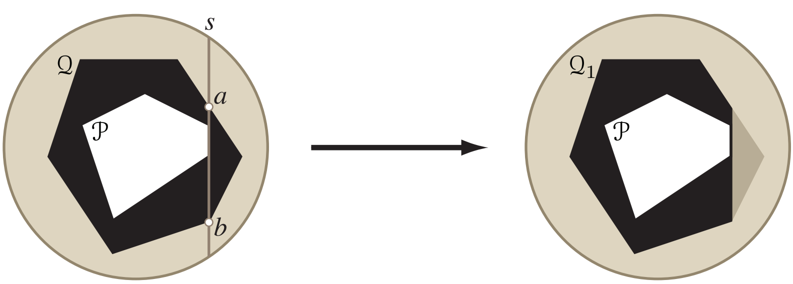
$$|\mathcal{P}| = \sum_{i=1}^{n} |P_i P_{i+1}|,$$

but I left it at that. In this lesson we are going to use perimeters of cyclic polygons to find the circumference of the circle. Along the way, I want to use the following result which compares the perimeters of two convex polygons when one is contained in the other.

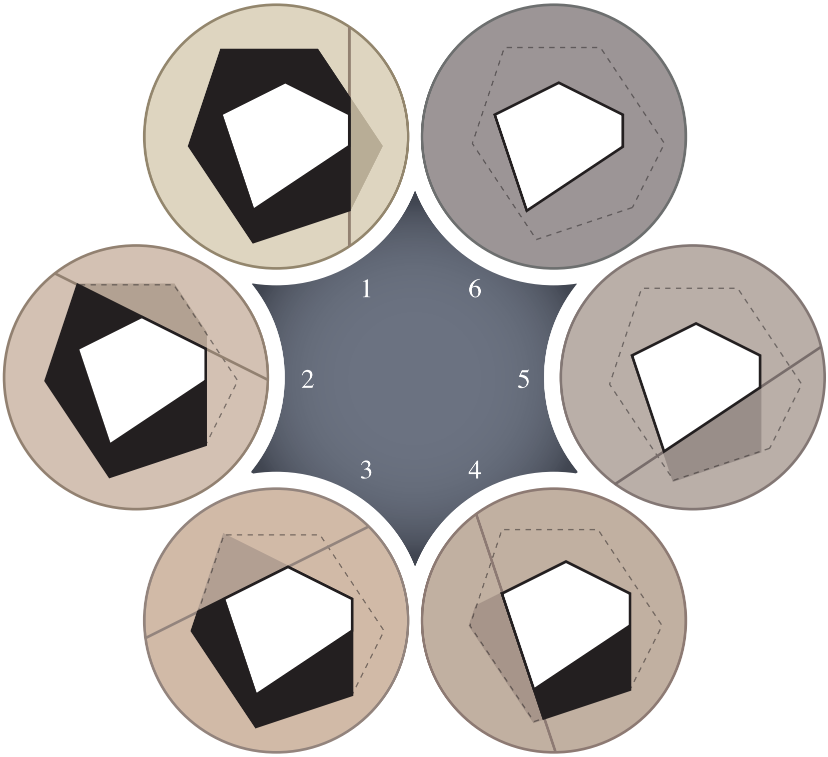
THM 1
If $\mathcal{P}$ and $\mathcal{Q}$ are convex polygons and all the points of $\mathcal{P}$ are on or inside $\mathcal{Q}$, then $|\mathcal{P}| \leq |\mathcal{Q}|$.

Proof. Some of the edges of $\mathcal{P}$ may run along the edges of $\mathcal{Q}$, but unless $\mathcal{P} = \mathcal{Q}$, at least one edge of $\mathcal{P}$ must pass through the interior of $\mathcal{Q}$. Let s be one of those interior edges. The line containing s intersects $\mathcal{Q}$ twice– call those intersections a and b– dividing $\mathcal{Q}$ into two smaller polygons which share the side ab, one on the same side of s as $\mathcal{P}$, the other on the opposite side. Essentially we want to "shave off" the part of $\mathcal{Q}$ on the opposite side, leaving behind only the polygon $\mathcal{Q}_1$ which consists of

- points of $\mathcal{Q}$ on the same side of s as $\mathcal{P}$, and
- points on the segment ab.



Shaving a polygon.



One at a time, shave the sides of the outer polygon down to the inner one.

There are two things to notice about $\mathcal{Q}_1$. First, $\mathcal{Q}_1$ and $\mathcal{P}$ have one more coincident side (the side s) than $\mathcal{Q}$ and $\mathcal{P}$ had. Second, the portions of $\mathcal{Q}$ and $\mathcal{Q}_1$ on the side of s with $\mathcal{P}$ are identical, so the segments making up that part contribute the same amount to their respective perimeters. On the other side, though, the path that $\mathcal{Q}$ takes from a to b is longer than the direct route along the segment ab of $\mathcal{Q}_1$ (because of the Triangle Inequality). Combining the two parts, that means $|\mathcal{Q}_1| \leq |\mathcal{Q}|$.

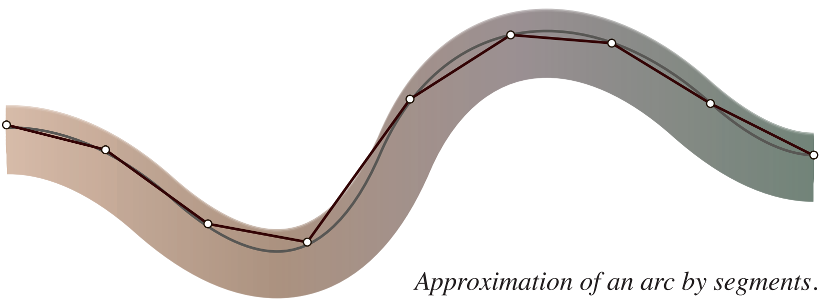
Now we can repeat this process with $\mathcal{P}$ and $\mathcal{Q}_1$, generating $\mathcal{Q}_2$ with even smaller perimeter than $\mathcal{Q}_1$ and another coincident side with $\mathcal{P}$. And again, to get $\mathcal{Q}_3$. Eventually, though, after say m steps, we run out of sides that pass through the interior, at which point $\mathcal{P} = \mathcal{Q}_m$. Then

$$|\mathcal{P}| = |\mathcal{Q}_m| \leq |\mathcal{Q}_{m-1}| \leq \cdots |\mathcal{Q}_2| \leq |\mathcal{Q}_1| \leq |\mathcal{Q}|.$$

□

Circumference

Geometers have drawn circles for a long time. I don't think it is a big surprise, then, that they would wonder about the relationship between the distance around the circle (how far they have dragged their pencil) and the radius of the circle. The purpose of this lesson is to answer that question. Our final result, the formula $C = 2\pi r$, sits right next to the Pythagorean Theorem in terms of star status, but I think it is a misunderstood celebrity. So let me be clear about what this equation is *not*. It is *not* an equation comparing two known quantities C and $2\pi r$. Instead, this equation is the way that we define the constant π. Nevertheless, the equation is saying *something* about the relationship between C and r– it is saying that the ratio of the two is a constant.



Approximation of an arc by segments.

To define the circumference of a circle, I want to take an idea from calculus– the idea of approximating a curve by straight line segments, and then refining the approximation by increasing the number of segments. In the case of a circle $\mathcal{C}$, the approximating line segments will be the edges of a simple cyclic polygon $\mathcal{P}$ inscribed in the circle. Conceptually, we will want the circumference of $\mathcal{C}$ to be bigger than the perimeter of $\mathcal{P}$. We should also expect that by adding in additional vertices to $\mathcal{P}$, we should be able to get the perimeter of $\mathcal{P}$ as close as we want to the circumference of $\mathcal{C}$. All this suggests (to me at least) that to get the circumference of $\mathcal{C}$, we need to find out how large the perimeters of inscribed polygons can be.

DEF: CIRCUMFERENCE

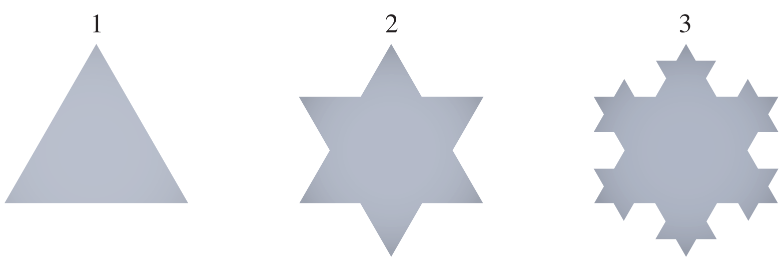
The circumference of a circle $\mathcal{C}$, written $|\mathcal{C}|$, is

$$|\mathcal{C}| = \sup\left\{ |\mathcal{P}| \,\middle|\, \mathcal{P} \text{ is a simple cyclic polygon inscribed in } \mathcal{C} \right\}.$$

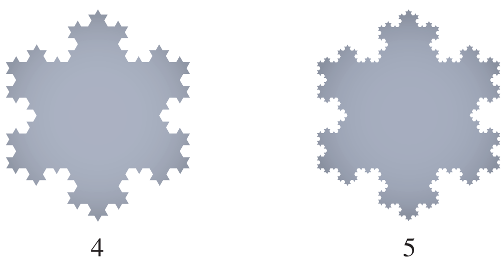
There is nothing in the definition to guarantee that this supremum exists. It is conceivable that the lengths of these approximating perimeters might just grow and grow with bound. One example of such degeneracy is given the deceptively cute name of “the Koch snowflake.” Let me describe how it works. Take an equilateral triangle with sides of length one. The perimeter of this triangle is, of course, 3. Now divide each of those sides into thirds. On each middle third, build an equilateral triangle by adding two more sides; then remove the the original side. You have made a shape with $3 \cdot 4$ sides, each with a length 1/3, for a perimeter of 4. Now iterate– divide each of those sides into thirds; build equilateral triangles on each middle third, and remove the base. That will make $3 \cdot 16$ sides of length 1/9, for a perimeter of 16/3. Then $3 \cdot 64$ sides of length 1/27 for a perimeter of $64/9$. Generally, after n iterations, there are $3 \cdot 4^n$ sides of length $1/3^n$ for a total perimeter of $4^n/3^{n-1}$, and

$$\lim_{n\to\infty} \frac{4^n}{3^{n-1}} = \lim_{n\to\infty} 3\left(\frac{4}{3}\right)^n = \infty.$$

The Koch snowflake, which is the limiting shape in this process, has infinite perimeter! The first thing we need to do, then, is to make sure that circles are better behaved than this.

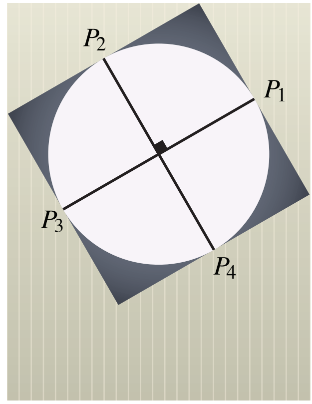


The first few steps in the construcion of the Koch snowflake.



AN UPPER BOUND FOR CIRCUMFERENCE

If $\mathcal{C}$ is a circle of radius r, then $|\mathcal{C}| \leq 8r$.



Proof. The first step is to build a circumscribing square around $\mathcal{C}$– the smallest possible square that still contains $\mathcal{C}$. Begin by choosing two perpendicular diameters d_1 and d_2. Each will intersect $\mathcal{C}$ twice, for a total of four intersections, P_1, P_2, P_3, and P_4. For each i between one and four, let t_i be the tangent line to $\mathcal{C}$ at P_i. These tangents intersect to form the circumscribing square. The length of each side of the square is equal to the diameter of $\mathcal{C}$, so the perimeter of the square is $4 \cdot 2r = 8r$.

Now we turn to the theorem we proved to start this lesson. Each simple cyclic polygon inscribed in $\mathcal{C}$ is a convex polygon contained in the circumscribing square. Therefore the perimeter of any such approximating polygon is bounded above by $8r$. Remember that we have defined $|\mathcal{C}|$ to be the supremum of all of these approximating perimeters, so it cannot exceed $8r$ either. □

Now that we know that any circle does have a circumference, the next step is to find a way to calculate it. The key to that is the next theorem.

CIRCUMFERENCE/RADIUS

The ratio of the circumference of a circle to its radius is a constant.

Proof. Let's suppose that this ratio is not a constant, so that there are two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with centers O_1 and O_2 and radii r_1 and r_2, but with unequal ratios

$$|\mathcal{C}_1|/r_1 > |\mathcal{C}_2|/r_2.$$

As we have defined circumference, there are approximating cyclic polygons to $\mathcal{C}_1$ whose perimeters are arbitrarily close to its circumference. In particular, there has to be some approximating cyclic polygon $\mathcal{P} = P_1P_2 \ldots P_n$ for $\mathcal{C}_1$ so that

$$|\mathcal{P}|/r_1 > |\mathcal{C}_2|/r_2.$$

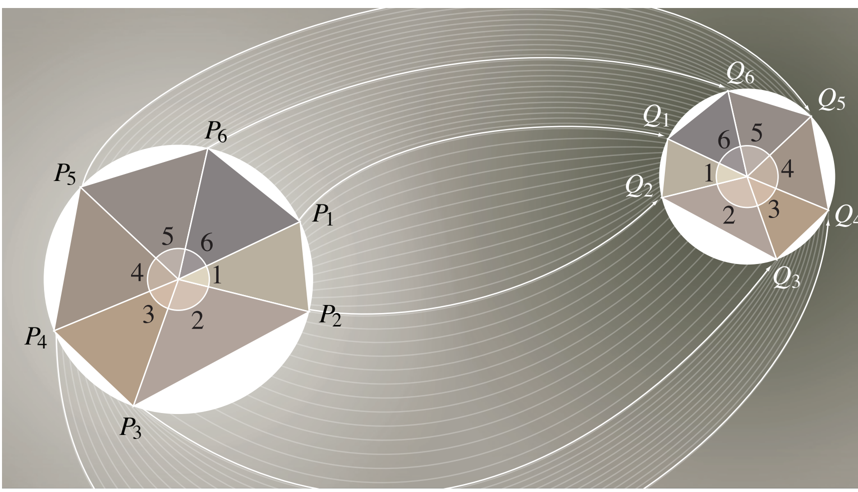
The heart of the contradiction is that we can build a cyclic polygon $\mathcal{Q}$ on $\mathcal{C}_2$ which is similar to $\mathcal{P}$ (intuitively, we just need to scale $\mathcal{P}$ so that it fits in the circle). The construction is as follows

1. Begin by placing a point Q_1 on circle $\mathcal{C}_2$.

2. Locate Q_2 on $\mathcal{C}_2$ so that $\angle P_1O_1P_2$ is congruent to $\angle Q_1O_2Q_2$ (there are two choices for this).

3. Locate Q_3 on $\mathcal{C}_2$ and on the opposite side of O_2Q_2 from Q_1 so that $\angle P_2O_1P_3 \simeq \angle Q_2O_2Q_3$.

4. Continue placing points on $\mathcal{C}_2$ in this fashion until Q_n has been placed to form the polygon $\mathcal{Q} = Q_1Q_2\ldots Q_n$.



Then

$$\frac{|O_2Q_i|}{|O_1P_i|} = \frac{r_2}{r_1} = \frac{|O_2Q_{i+1}|}{|O_1P_{i+1}|} \quad \& \quad \angle Q_iO_2Q_{i+1} \simeq \angle P_iO_1P_{i+1},$$

so by S·A·S similarity, $\triangle Q_iO_2Q_{i+1} \sim \triangle P_iO_1P_{i+1}$. That gives us the ratio of the third sides of the triangle as $|Q_iQ_{i+1}|/|P_iP_{i+1}| = r_2/r_1$ and so we can describe the perimeter of $\mathcal{Q}$ as

$$|\mathcal{Q}| = \sum_{i=1}^{n} |Q_iQ_{i+1}| = \sum_{i=1}^{n} \frac{r_2}{r_1}|P_iP_{i+1}| = \frac{r_2}{r_1}\sum_{i=1}^{n} |P_iP_{i+1}| = \frac{r_2}{r_1}|\mathcal{P}|.$$

Here's the problem. That would mean that

$$\frac{|\mathcal{Q}|}{r_2} = \frac{|\mathcal{P}|}{r_1} > \frac{|\mathcal{C}_2|}{r_2}$$

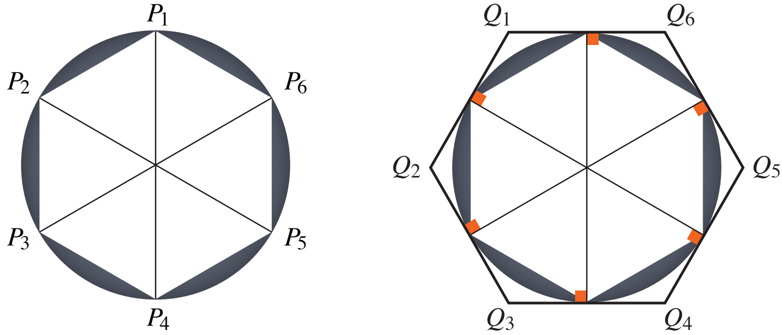
so $|\mathcal{Q}| > |\mathcal{C}_2|$ when the circumference of $\mathcal{C}_2$ is supposed to be greater than the perimeter of any of the approximating cyclic polygons. □

DEF: π

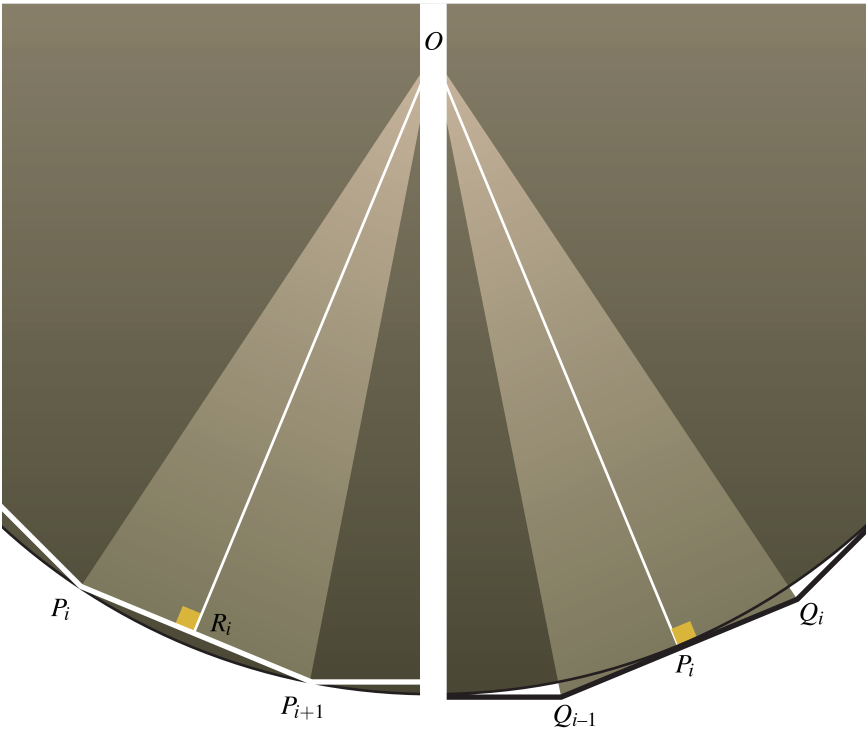
The constant π is the ratio of the circumference of a circle to its diameter

$$\pi = \frac{|\mathcal{C}|}{2r}.$$

The problem with this definition of circumference, and consequently this definition of π, is that it depends upon a supremum, and supremums are ungainly and difficult to maneuver. A limit is considerably more nimble. Fortunately, this particular supremum can be reached via the perimeters of a sequence of regular polygons as follows. Arrange n angles each measuring $360^\circ/n$ around the center of any circle $\mathcal{C}$. The rays of those angles intersect $\mathcal{C}$ n times, and these points P_i are the vertices of a regular n-gon, $\mathcal{P}_n = P_1P_2\ldots P_n$. The tangent lines to $\mathcal{C}$ at the neighboring points P_i and P_{i+1} intersect at a point Q_i. Taken together, these n points are the vertices of another regular n-gon $\mathcal{Q}_n = Q_1Q_2\ldots Q_n$. The polygon $\mathcal{P}_n$ is just one of the many cyclic polygons inscribed in $\mathcal{C}$ so $|\mathcal{P}_n| \le |\mathcal{C}|$. The polygon $\mathcal{Q}_n$ circumscribes $\mathcal{C}$, and every cyclic polygon inscribed on $\mathcal{C}$ lies inside $\mathcal{Q}_n$, so $|\mathcal{Q}_n| \ge |\mathcal{C}|$.



Regular inscribed and circumscribing hexagons.



The lower bound prescribed by $\mathcal{P}_n$. Each $OQ_i \rightarrow$ is a perpendicular bisector of P_iP_{i+1}, intersecting it at a point R_i and dividing $\triangle OP_iP_{i+1}$ in two. By the H·L congruence theorem for right triangles, those two parts, $\triangle OR_iP_i$ and $\triangle OR_iP_{i+1}$, are congruent. That means that $\mathcal{P}_n$ is built from $2n$ segments of length $|P_iR_i|$. Now

$$\sin(360^\circ/2n) = \frac{|P_iR_i|}{r}$$
$$\implies |P_iR_i| = r\sin(360^\circ/2n)$$

so

$$|\mathcal{P}_n| = 2nr\sin(360^\circ/2n).$$

The upper bound prescribed by $\mathcal{Q}_n$. Each $OP_i \rightarrow$ is a perpendicular bisector of $Q_{i-1}Q_i$, intersecting it at P_i and dividing $\triangle OQ_{i-1}Q_i$ in two. By S·A·S, the two parts, $\triangle OP_iQ_{i-1}$ and $\triangle OP_iQ_i$, are congruent. That means $\mathcal{Q}_n$ is built from $2n$ segments of length $|P_iQ_i|$. Now

$$\tan(360^\circ/2n) = |P_iQ_i|/r$$
$$\implies |P_iQ_i| = r\tan(360^\circ/2n)$$

so

$$|\mathcal{Q}_n| = 2nr\tan(360^\circ/2n).$$

Let's compare $|\mathcal{P}_n|$ and $|\mathcal{Q}_n|$ as n increases (the key to this calculation is that as x approaches zero, $\cos(x)$ approaches one):

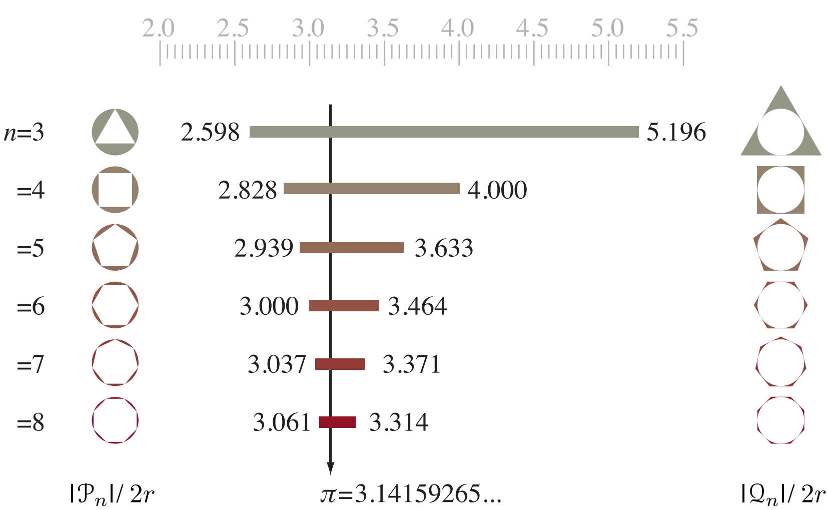
$$\begin{aligned}
\lim_{n\to\infty} |\mathcal{Q}_n| &= \lim_{n\to\infty} 2nr\tan(360^\circ/2n) \\
&= \lim_{n\to\infty} \frac{2nr\sin(360^\circ/2n)}{\cos(360^\circ/2n)} \\
&= \frac{\lim_{n\to\infty} 2nr\sin(360^\circ/2n)}{\lim_{n\to\infty}\cos(360^\circ/2n)} \\
&= \lim_{n\to\infty} 2nr\sin(360^\circ/2n)/1 \\
&= \lim_{n\to\infty} |\mathcal{P}_n|.
\end{aligned}$$

Since $|\mathcal{C}|$ is trapped between $|\mathcal{P}_n|$ and $|\mathcal{Q}_n|$ for all n, and since these are closing in upon the same number as n goes to infinity, $|\mathcal{C}|$ must also be approaching this number. That gives a more comfortable equation for circumference as

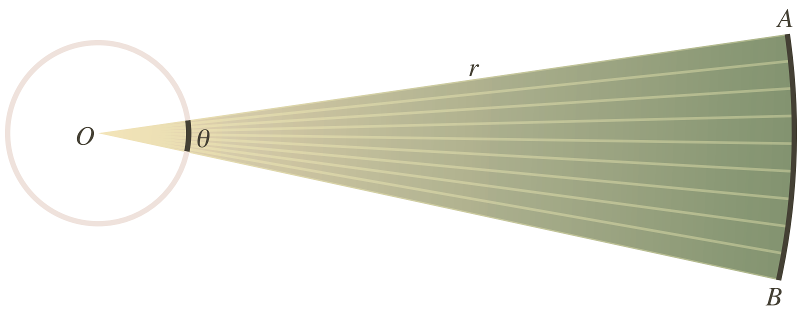
$$|\mathcal{C}| = \lim_{n\to\infty} 2nr\sin(360^\circ/2n),$$

and since $|\mathcal{C}| = 2\pi r$, we can disentangle a definition of π as

$$\pi = \lim_{n\to\infty} n\sin(360^\circ/2n).$$



Upper and lower bounds for π.



Lengths of arcs and radians.

It doesn't take much modification to get a formula for a length of arc. The 360° in the formula for $|\mathcal{C}|$ is the measure of the central angle corresponding to an arc that goes completely around the circle. To get the measure of any other arc, we just need to replace the 360° with the measure of the corresponding central angle.

LENGTHS OF CIRCULAR ARCS
If $\smile AB$ is the arc of a circle with radius r, and if θ is the measure of the central angle $\angle AOB$, then

$$|\smile AB| = \frac{\pi}{180^\circ}\theta \cdot r.$$

Proof. To start, replace the 360° in the circumference formula with θ:

$$|\smile AB| = \lim_{n\to\infty} 2nr\sin(\theta/2n) = 2r \cdot \lim_{n\to\infty} n\sin(\theta/2n).$$

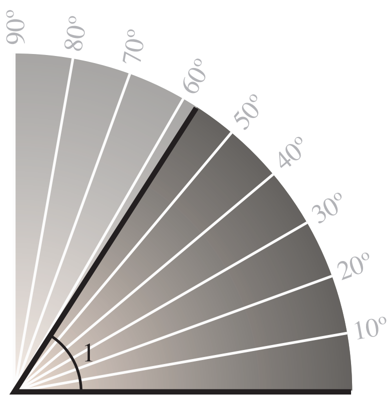
This limit is clearly related to the one that defines π. I want to absorb the difference between the two into the variable via the substitution $n = m \cdot \theta/360^\circ$. Note that as n approaches infinity, m will as well, so

$$\begin{aligned}|\smile AB| &= 2r \cdot \lim_{m\to\infty} \frac{m\cdot\theta}{360^\circ} \sin\left(\frac{\theta}{2m\cdot\theta/360^\circ}\right) \\ &= \frac{2r\theta}{360^\circ} \cdot \lim_{m\to\infty} m\sin(360^\circ/2m) \\ &= \frac{\theta}{180^\circ} r\pi.\end{aligned}$$

$\square$

There is one more thing to notice before the end of this lesson. This arc length formula provides a most direct connection between angle measure (of the central angle) and distance (along the arc). And yet, the $\frac{\pi}{180^\circ}$ factor in that formula suggests that distance and the degree measurement system are a little out of sync with one another. This can be fixed by modernizing our method of angle measurement. The preferred angle measurement system, and the one that I will use from here on out, is *radian* measurement.

DEF: RADIAN
One radian is $\pi/180^\circ$.



One radian is approximately 57.296°.

The measure of a straight angle is π radians. The measure of a right angle is $\pi/2$ radians. One complete turn of the circle is 2π radians. If $\theta = (\angle AOB)$ is measured in radians, then

$$|\smile AB| = r \cdot \theta.$$

References

The Koch snowflake is an example of a fractal. Gerald Edgar's book *Measure, Topology, and Fractal Geometry* [1], deals with these objects and their measures.

[1] Gerald A. Edgar. *Measure, Topology, and Fractal Geometry*. Springer-Verlag, New York, 1st edition, 1990.

Exercises

1. Let A and B be points on a circle $\mathcal{C}$ with radius r. Let θ be the measure of the central angle corresponding to the minor arc (or semicircle) $\smile AB$. What is the relationship (in the form of an equation) between θ, r, and $|AB|$?

2. Let AB be a diameter of a circle $\mathcal{C}$, and let P be a point on AB. Let $\mathcal{C}_1$ be the circle with diameter AP and let $\mathcal{C}_2$ be the circle with diameter BP. Show that the sum of the circumferences of $\mathcal{C}_1$ and $\mathcal{C}_2$ is equal to the circumference of $\mathcal{C}$ (the shape formed by the three semicircles on one side of AB is called an *arbelos*).

3. In the construction of the Koch snowflake, the middle third of each segment is replaced with two-thirds of an equilateral triangle. Suppose, instead, that middle third was replaced with three of the four sides of a square. What is the perimeter of the n-th stage of this operation? Would the limiting perimeter still be infinite?

4. This problem deals with the possibility of angle measurement systems other than degrees or radians. Let $\mathcal{A}$ be the set of angles in the plane. Consider a function

$$\star : \mathcal{A} \to (0,\infty) : \angle A \to (\angle A)^\star$$

which satisfies the following properties

(1) if $\angle A \simeq \angle B$, then $(\angle A)^\star = (\angle B)^\star$

(2) if D is in the interior of $\angle ABC$, then

$$(\angle ABC)^\star = (\angle ABD)^\star + (\angle DBC)^\star.$$

Prove that the $\star$ measurement system is a constant multiple of the degree measurement system (or, for that matter, the radian measurement system). That is, prove that there is a $k > 0$ such that for all $\angle A \in \mathcal{A}$,

$$(\angle A)^\star = k \cdot (\angle A).$$