1 NEUTRAL GEOMETRY

The goal of this book is to provide a pleasant but thorough introduction to Euclidean and non-Euclidean (hyperbolic) geometry. Before I go any further, let me clear up something that could lead to confusion on down the road. Some mathematicians use the term *non-Euclidean geometry* to mean any of a whole host of geometries which fail to be Euclidean for any number of reasons. The kind of non-Euclidean geometry that we will study in these lessons, and the kind that I mean when I use the term *non-Euclidean geometry*, is something much more specific– it is a geometry that satisfies all of Hilbert's axioms for Euclidean geometry except the parallel axiom.

It turns out that that parallel axiom is absolutely central to the nature of the geometry. The Euclidean geometry with the parallel axiom and the non-Euclidean geometry without it are radically different. Even so, Euclidean and non-Euclidean geometry are not polar opposites. As different as they are in many ways, they still share many basic characteristics. Neutral geometry (also known as absolute geometry in older texts) is the study of those commonalities.

1. OUR DUCKS IN A ROW

THE AXIOMS OF INCIDENCE AND ORDER

From Euclid to Hilbert

You pretty much have to begin a study of Euclidean geometry with at least some mention of Euclid's *Elements*, the book that got the ball rolling over two thousand years ago. *The Elements* opens with a short list of definitions. As discussed in the previous chapter, the first few of these definitions are a little problematic. If we can push past those, we get to Euclid's five postulates, the core accepted premises of his development of the subject.

EUCLID'S POSTULATES

P1 To draw a straight line from any point to any point.

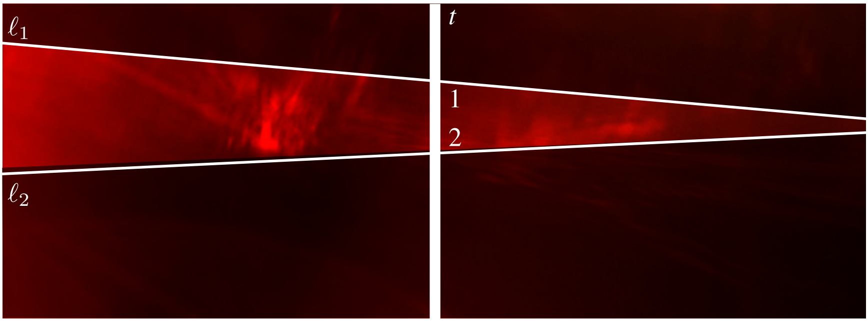
P2 To produce a finite straight line continuously in a straight line.

P3 To describe a circle with any center and distance.

P4 That all right angles are equal to one another.

P5 That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The first three postulates describe constructions. Today we would probably reinterpret them as statements about the existence of certain objects. The fourth provides a way to compare angles. As for the fifth, well, in all of history, not many sentences have received as much scrutiny as that one.



Euclid's Parallel Postulate

Because $(\angle 1) + (\angle 2) < 180°$, ℓ_1 *and* ℓ_2 *intersect on this side of* t.

When you look at these postulates, and Euclid's subsequent development of the subject from them, it appears that Euclid may have been attempting an axiomatic development of the subject. There is some debate, though, about the extent to which Euclid really was trying to do that. His handling of "S·A·S," for example, is not founded upon the postulates, and not merely in a way that might be attributed to oversight. With a couple thousand years between us and him, we can only guess at his true intentions. In any case, Euclidean geometry was not properly and completely axiomatized until much later, at the end of the nineteenth century by the German mathematician David Hilbert. His 1899 book, *The Foundations of Geometry* gave an axiomatic description of what we think of as Euclidean geometry. Subsequently, there have been several other axiomatizations, including notably ones by Birkhoff and Tarski. The nice thing about Hilbert's approach is that proofs developed in his system "feel" like Euclid's proofs. Some of the other axiomatizations, while more streamlined, do not retain that same feel.

Neutral Geometry

It might be an obvious statement, but it needs to be said: Euclid's Fifth Postulate does not look like the other four. It is considerably longer and more convoluted than the others. For that reason, generations of geometers after Euclid hoped that the Fifth might actually be provable– that it could be taken as a theorem rather than a postulate. From their efforts (which, by the way, were unsuccessful) there arose a whole area of study. Called *neutral geometry* or *absolute geometry*, it is the study of the geometry of the plane without Euclid's Fifth Postulate.

So what exactly do you give up when you decide not to use Euclid's Fifth? Essentially Euclid's Fifth tells us something about the nature of parallel lines. It does so in a rather indirect way, though. Nowadays it is common to use Playfair's Axiom in place of Euclid's Fifth because it addresses the issue of parallels much more directly. Playfair's Axiom both implies and is implied by Euclid's Fifth, so the two statements can be used interchangeably.

PLAYFAIR'S AXIOM
For any line ℓ and for any point P which is not on ℓ, there is exactly one line through P which is parallel to ℓ.

Even without Playfair's Axiom, it is relatively easy to show that there must be at *least* one parallel through P, so what Playfair's Axiom is really telling us is that in Euclidean geometry there cannot be *more* than one parallel. The existence of a unique parallel is crucial to many of the proofs of Euclidean geometry. Without it, neutral geometry is quite limited. Still, neutral geometry is the common ground between Euclidean and non-Euclidean geometries, and it is where we begin our study.

In the first part of this book, we are going to develop neutral geometry following the approach of Hilbert. In Hilbert's system there are five undefined terms: *point*, *line*, *on*, *between*, and *congruent*. Fifteen of his axioms are needed to develop neutral plane geometry. Generally the axioms are grouped into categories to make it a bit easier to keep track of them: the axioms of incidence, the axioms of order, the axioms of congruence, and the axioms of continuity. We will investigate them in that order over the next several chapters.

Incidence

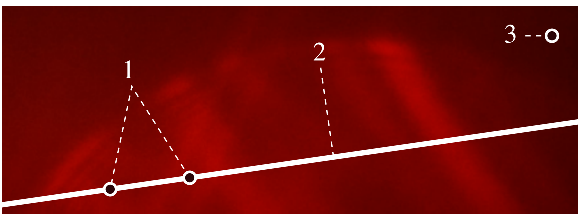
Hilbert's first set of axioms, the axioms of incidence, describe the interaction between points and lines provided by the term *on*. *On* is a binary relationship between points and lines so, for instance, you can say that a point P is (or is not) on a line ℓ. In situations where you want to express the line's relationship to a point, rather than saying that a line ℓ is on a point P (which is technically correct), it is much more common to say that ℓ passes through P.

THE AXIOMS OF INCIDENCE

In 1 There is a unique line on any two distinct points.

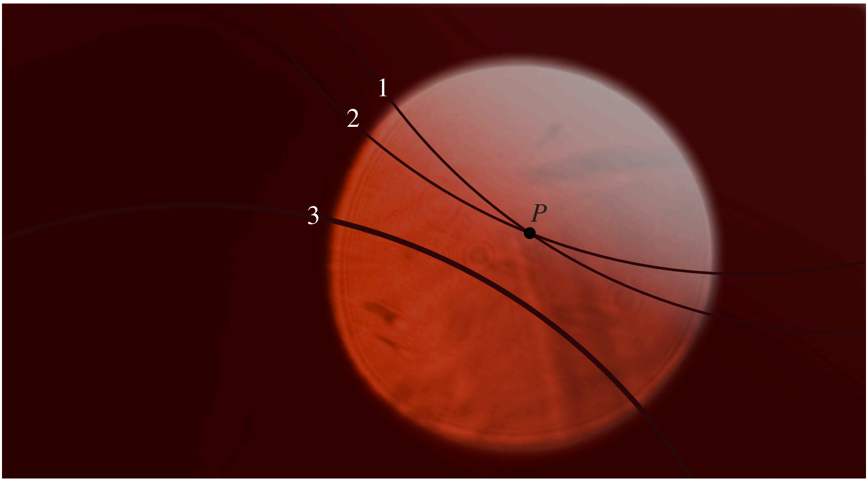
In 2 There are at least two points on any line.

In 3 There exist at least three points that do not all lie on the same line.



Incidence

1 Two points on a line
2 A line on two points
3 And there's more.



Lines 1 and 2 intersect. Both are parallel to line 3. Because there appear to be two lines through P parallel to line 3, this does not look like Euclidean geometry.

By themselves, the axioms of incidence do not afford a great wealth of theorems. Some notation and a few definitions are all we get. First, the notation. Because of the first axiom, there is only one line through any two distinct points. Therefore, for any two distinct points A and B, we use the notation $\leftarrow AB \rightarrow$ to denote the line through A and B. As you are probably all aware, this is not exactly the standard notation for a line. Conventionally, the line symbol is placed above the points. I just don't like that notation in print– unless you have lots of room between lines of text, the symbol crowds the line above it.

Now the definitions. Any two distinct points lie on one line. Three or more points may or may not all lie on the same line.

DEF: COLINEARITY

Three or more points are *colinear* if they are all on the same line and are *non-colinear* if they are not.

According to the first axiom, two lines can share at most one point. However, they may not share any points at all.

DEF: PARALLEL AND INTERSECTING

Two lines *intersect* if there is a point P which is on both of them. In this case, P is the *intersection* or *point of intersection* of them. Two lines which do not share a point are *parallel*.

Order

The axioms of order describe the undefined term *between*. Between is a relation between a point and a pair of points. We say that a point B is, or is not, between two points A and C and we use the notation $A*B*C$ to indicate that B is between A and C. Closely related to this "between-ness" is the idea that a line separates the plane. This behavior, which is explained in the last of the order axioms, depends upon the following definition.

DEF: SAME SIDE

Let ℓ be a line and let A and B be two points which are not on ℓ. Points A and B are on the *same side* of ℓ if either ℓ and $\overleftrightarrow{AB}$ do not intersect at all, or if do they intersect but the point of intersection is not between A and B.

So now, without further delay, the Axioms of Order describing the properties of between.

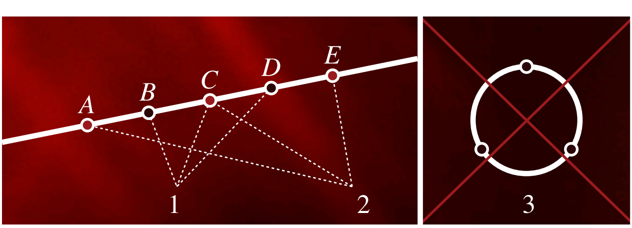
THE AXIOMS OF ORDER

Or 1 If $A*B*C$, then the points A, B, C are distinct colinear points, and $C*B*A$.

Or 2 For any two points B and D, there are points A, C, and E, such that $A*B*D$, $B*C*D$ and $B*D*E$.

Or 3 Of any three distinct points on a line, exactly one lies between the other two.

Or 4 *The Plane Separation Axiom.* For any line ℓ and points A, B, and C which are not on ℓ: (i) If A and B are on the same side of ℓ and A and C are on the same side of ℓ, then B and C are on the same side of ℓ. (ii) If A and B are not on the same side of ℓ and A and C are not on the same side of ℓ, then B and C are on the same side of ℓ.



Order

1 Points in order
2 Between and beyond
3 But no circularity

The last of these, the Plane Separation Axiom (*PSA*), is a bit more to digest than the previous axioms. It is pretty critical though– it is the axiom which limits plane geometry to two dimensions. Let's take a closer look. Let ℓ be a line and let P be a point which is not on ℓ. We're going to define two sets of points.

> S_1: P itself and all points on the same side of ℓ as P.
>
> S_2: all points which are not on ℓ nor on the same side of ℓ as P

By the second axiom of order both S_1 and S_2 are nonempty sets. The first part of *PSA* tells us is that all the points of S_1 are on the same side; the second part tells us that all the points of S_2 are on the same side. Hence there are two and only two sides to a line. Because of this, we can refer to points which are not on the same side of a line as being on *opposite sides*.

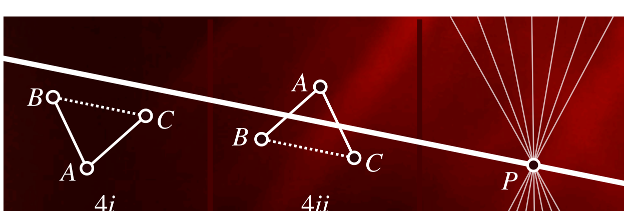
Just as a line separates the remaining points of the plane, a point on a line separates the remaining points on that line. If P is between A and B, then A and B are on *opposite sides* of P. Otherwise, A and B are on the *same side* of P. You might call this separation of a line by a point "line separation". It is a direct descendent of plane separation via the following simple correspondence. For three distinct points A, B, and P on a line ℓ,

A, B on the same side of P	$\iff$	A, B are on the same side of any line through P other than ℓ
A, B on opposite sides of P	$\iff$	A, B are on opposite sides of any line through P other than ℓ

Because of this, there is a counterpart to the Plane Separation Axiom for lines. Suppose that A, B, C and P are all on a line. (1) If A and B are on the same side of P and A and C are on the same side of P, then B and C are on the same side of P. (2) If A and B are on opposite sides of P and A and C are on opposite sides of P, then B and C are on the same side of P. As a result, a point divides a line into *two* sides.

PSA

A line separates the plane. A point separates a line.



With *between*, we can now introduce some a few of the main characters in this subject.

DEF: LINE SEGMENT
For any two points A and B, the *line segment* between A and B is the set of points P such that $A * P * B$, together with A and B themselves. The points A and B are called the *endpoints* of the segment.

DEF: RAY
For two distinct points A and B, the *ray* from A through B consists of the point A together with all the points on $\leftarrow AB \rightarrow$ which are on the same side of A as B. The point A is called the *endpoint* of the ray.

The notation for the line segment between A and B is AB. For rays, I write $AB\rightarrow$ for the ray with endpoint A through the point B. As with my notation for lines, this is a break from the standard notation which places the ray symbol above the letters.

DEF: OPPOSITE RAY
For any ray $AB\rightarrow$, the opposite ray $(AB\rightarrow)^{op}$ consists of the point A together with all the points of $\leftarrow AB \rightarrow$ which are on the opposite side of A from B.

Putting Points in Order

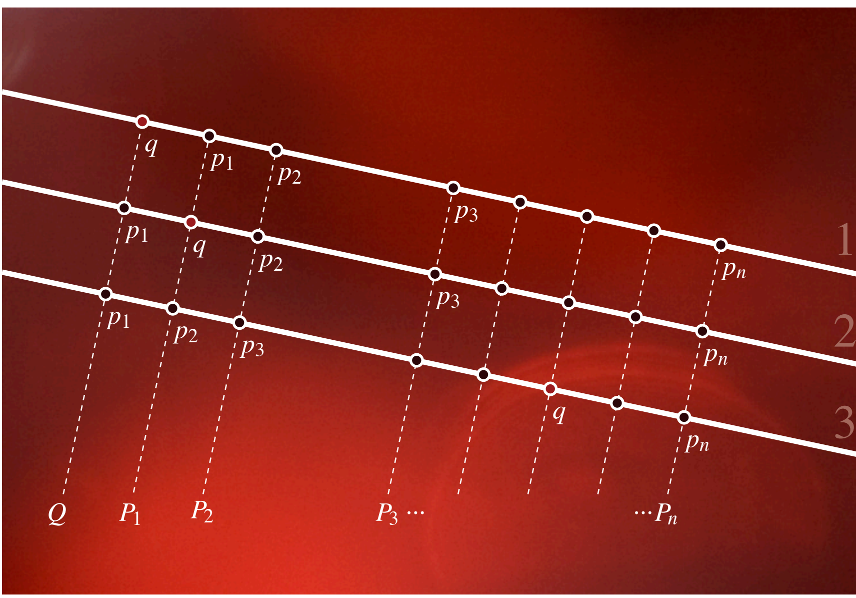
The order axioms describe how to put three points in order. Sometimes, though, three is not enough. It would be nice to know that more than three points on a line can be ordered in a consistent way. Thankfully, the axioms of order make this possible as well.

THM: ORDERING POINTS
Given $n \geq 3$ colinear points, there is a labeling of them $P_1, P_2, \ldots, P_n$ so that if $1 \leq i < j < k \leq n$, then $P_i * P_j * P_k$. In that case, we write

$$P_1 * P_2 * \cdots * P_n.$$

Proof. This is a proof by induction. The initial case, when there are just three points to put in order, is an immediate consequence of the axioms of order. Now let's assume that any set of n colinear points can be put in order, and let's suppose we want to put a set of $n+1$ colinear points in order. I think the natural way to do this is to isolate the first point (call it Q), put the remaining points in order, and then stick Q back on the front. The problem with this approach is that figuring out which point is the first point essentially presupposes that you *can* put the points in order. Getting around this is a little delicate, but here's how it works. Choose n of the $n+1$ points. Put them in order and label them so that $p_1 * p_2 * \cdots * p_n$. Let q be the one remaining point. Now, one of the following three things must happen:

$$q * p_1 * p_2 \quad \text{or} \quad p_1 * q * p_2 \quad \text{or} \quad p_1 * p_2 * q.$$

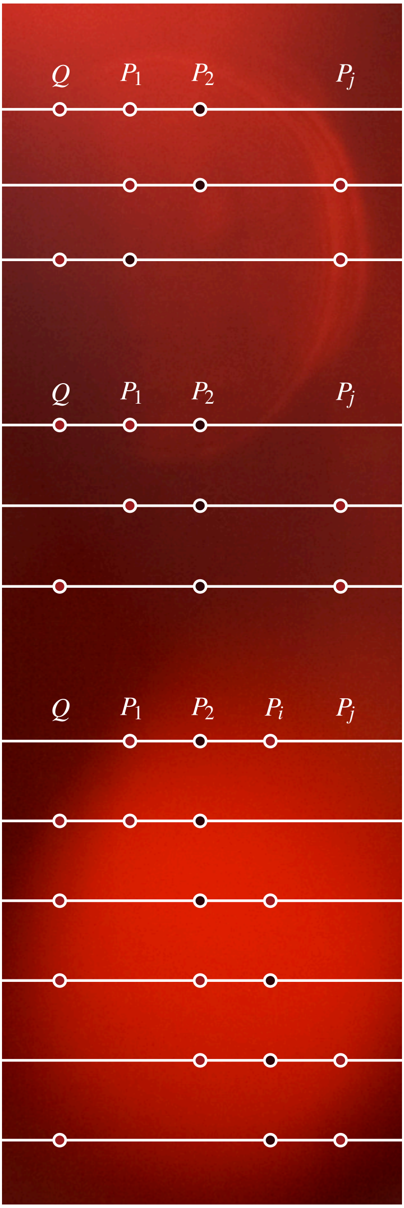


The three possible positions of q in relation to p_1 and p_2.

In the first case, let $Q = q$ and let $P_1 = p_1, P_2 = p_2, \ldots, P_n = p_n$. In the second and third cases, let $Q = p_1$.Then put the remaining points $p_1, \ldots, p_n$ and q in order and label them $P_1, P_2, \ldots, P_n$. Having done this, we have two pieces of an ordering

$$Q * P_1 * P_2 \quad \text{and} \quad P_1 * P_2 * \cdots * P_n.$$

The proof is not yet complete, though, because we still need to show that Q is ordered properly with respect to the remaining P's. That is, we need to show $Q * P_i * P_j$ when $1 \leq i < j \leq n$. Let's do that (in several cases).



Case 1: $i = 1$.
The result is given when $j = 2$, so let's suppose that $j > 2$. Then:

1. $Q * P_1 * P_2$ so Q and P_1 are on the same side of P_2.

2. $P_1 * P_2 * P_j$ so P_1 and P_j are on opposite sides of P_2.

3. Therefore Q and P_j are on opposite sides of P_1, so $Q * P_1 * P_j$.

Case 2: $i = 2$.

1. $Q * P_1 * P_2$ so Q and P_1 are on the same side of P_2.

2. $P_1 * P_2 * P_j$ so P_1 and P_j are on opposite sides of P_2.

3. Therefore Q and P_j are on opposite sides of P_2, so $Q * P_2 * P_j$.

Case 3: $i > 2$.

1. $P_1 * P_2 * P_i$ so P_1 and P_i are on opposite sides of P_2.

2. $Q * P_1 * P_2$ so Q and P_1 are on the same side of P_2.

3. Therefore Q and P_i are on opposite sides of P_2, so $Q * P_2 * P_i$.

4. Consequently, Q and P_2 are on the same side of P_i.

5. Meanwhile, $P_2 * P_i * P_j$ so P_2 and P_j are on opposite sides of P_i.

6. Therefore, Q and P_j are on opposite sides of P_i, so $Q * P_i * P_j$.

$\square$

Exercises

1. Prove that if $A*B*C$ then $AB \subset AC$ and $AB{\rightarrow} \subset AC{\rightarrow}$.

2. Prove that if $A*B*C*D$ then $AC \cup BD = AD$ and $AC \cap BD = BD$.

3. Prove that the points which are on both $AB\rightarrow$ and $BA\rightarrow$ are the points of AB.

4. Use the axioms of order to show that there are infinitely many points on any line and that there are infinitely many lines through a point.

5. The familiar model for Euclidean geometry is the "Cartesian model." In that model, points are interepreted as coordinate pairs of real numbers (x,y). Lines are loosely interpreted as equations of the form

$$Ax+By=C$$

but technically, there is a little bit more to it than that. First, A and B cannot both simultaneously be zero. Second, if $A'=kA$, $B'=kB$, and $C'=kC$ for some nonzero constant k, then the equations $Ax+By=C$ and $A'x+B'y=C'$ both represent the same line [in truth then, a line is represented by an equivalence class of equations]. In this model, a point (x,y) is on a line $Ax+By=C$ if its coordinates make the equation true. With this interpretation, verify the axioms of incidence.

6. In the Cartesian model, a point (x_2,y_2) is between two other points (x_1,y_1) and (x_3,y_3) if:

   1. the three points are distinct and on the same line, and
   2. x_2 is between x_1 and x_3 (either $x_1 \leq x_2 \leq x_3$ or $x_1 \geq x_2 \geq x_3$), and
   3. y_2 is between y_1 and y_3 (either $y_1 \leq y_2 \leq y_3$ or $y_1 \geq y_2 \geq y_3$).

   With this interpretation, verify the axioms of order.

Further reading

For these first few"moves", we are pretty constricted, with few results to build from and very little flexibility about where we can go next. Since we have adopted the axioms of Hilbert, our initial steps (in this and the next few lessons) follow fairly closely those of Hilbert in his *Foundations of Geometry* [2].

In addition, let me refer you to a few more contemporary books which examine the first steps in the development of the subject. Moise's *Elementary Geometry from an Advanced Standpoint* [3] is one of my favorites. I have taught from both Wallace and West's *Roads to Geometry* [4], and Greenberg's *Euclidean and Non-Euclidean Geometries* [1].

[1] Marvin J. Greenberg. *Euclidean and Non-Euclidean Geometries: Development and History*. W.H. Freeman and Company, New York, 4th edition, 2008.

[2] David Hilbert. *The Foundations of Geometry*.

[3] Edwin E. Moise. *Elementary Geometry from an Advanced Standpoint*. Addison Wesley Publishing Company, Reading, Massachusetts, 2nd edition, 1974.

[4] Edward C. Wallace and Stephen F. West. *Roads to Geometry*. Pearson Education, Inc., Upper Saddle River, New Jersey, 3rd edition, 2004.