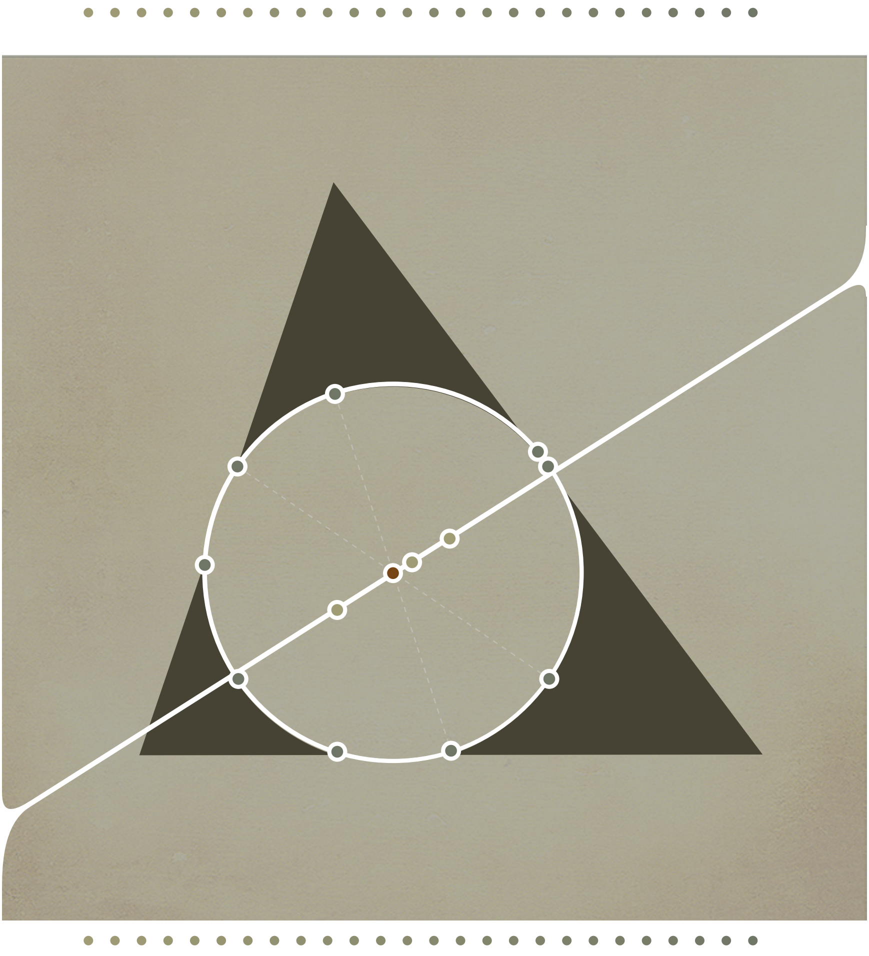
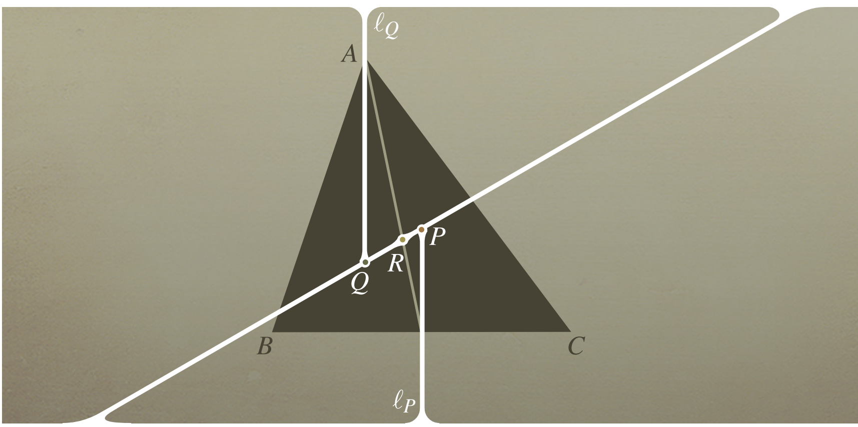


20 CONCURRENCE II

The Euler line

I wrapped up the last lesson with illustrations of three triangles and their centers, but I wonder if you noticed something in those illustrations? In each one, it certainly appears that the circumcenter, orthocenter, and centroid are colinear. Well, guess what– this is no coincidence.

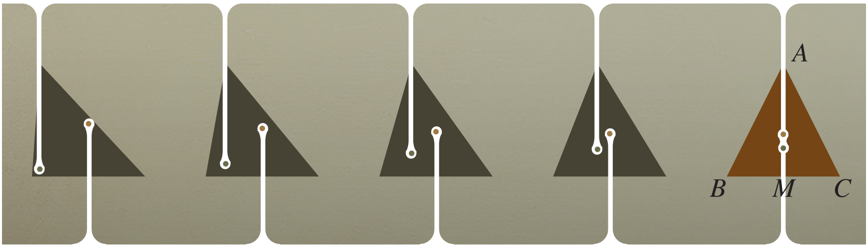
THM: THE EULER LINE
The circumcenter, orthocenter and centroid of a triangle are colinear, on a line called the *Euler line*.



Proof. First, the labels. On $\triangle ABC$, label

P: the circumcenter
Q: the orthocenter
R: the centroid
M: the midpoint of BC
ℓ_P: the perpendicular bisector to BC
ℓ_Q: the altitude through A
ℓ_R: the line containing the median AM

A dynamic sketch of all these points and lines will definitely give you a better sense of how they interact. Moving the vertices A, B, and C creates a rather intricate dance of P, Q and R. One of the most readily apparent features of this construction is that both ℓ_P and ℓ_Q are perpendicular to BC, and that means they cannot intersect unless they coincide. If you do have a sketch to play with, you will see that they *can* coincide.



Aligning an altitude and a perpendicular bisector.

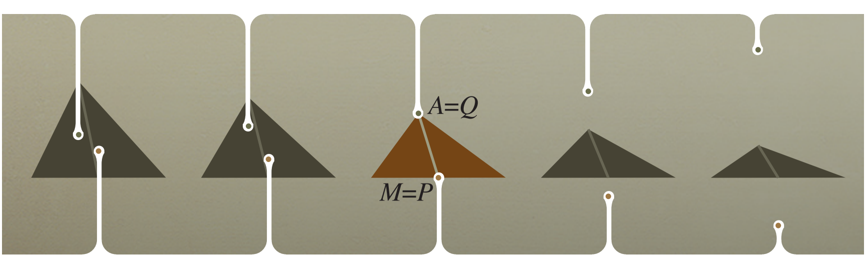
This is a good place to start the investigation.

$$\begin{aligned}
&\ell_P = \ell_Q \\
\iff\ &\ell_R \text{ intersects } BC \text{ at a right angle} \\
\iff\ &\triangle AMB \text{ is congruent to } \triangle AMC \\
\iff\ &AB \simeq AC
\end{aligned}$$

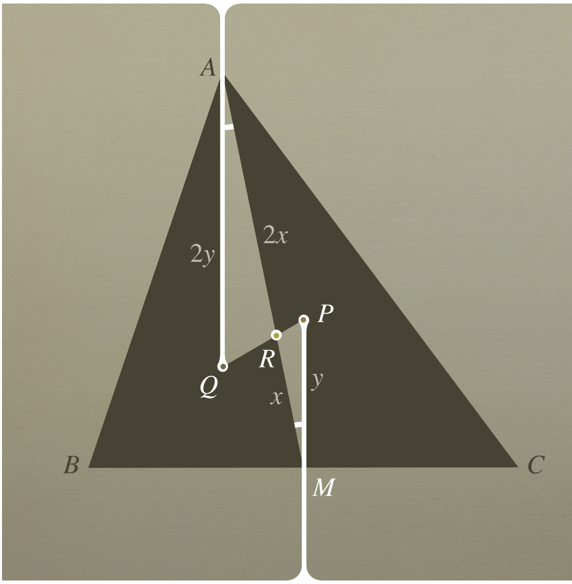
So in an isosceles triangle with congruent sides AB and AC, all three of P and Q and R will lie on the line $\ell_P = \ell_Q = \ell_R$. It is still possible to line up P, Q and R along the median AM without having ℓ_P, ℓ_Q and ℓ_R coincide. That's because ℓ_P intersects AM at M and ℓ_Q intersects AM at A, and it turns out that it is possible to place P at M and Q at A.

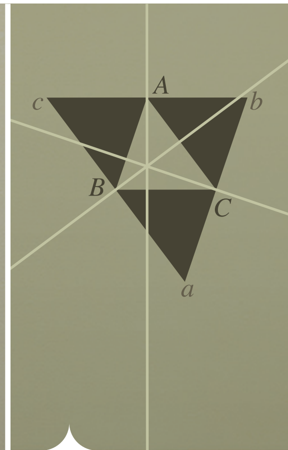
$$\begin{aligned}
&M \text{ is the circumcenter} \\
\iff\ &BC \text{ is a diameter of the circumcircle} \\
\iff\ &\angle A \text{ is a right angle (Thales' theorem)} \\
\iff\ &AB \text{ and } AC \text{ are both altitudes of } \triangle ABC \\
\iff\ &A \text{ is the orthocenter}
\end{aligned}$$

So if $\triangle ABC$ is a right triangle whose right angle is at vertex A, then again the median AM contains P, Q, and R.



Putting the circumcenter and orthocenter on a median.





The second S. One triangle's altitudes are another triangle's perpendicular bisectors.

In all other scenarios, P and Q will *not* be found on the median, and this is where things get interesting. At the heart of this proof are two triangles, $\triangle AQR$ and $\triangle MPR$. We must show they are similar.

S: We saw in the last lesson that the centroid is located two thirds of the way down the median AM from A, so $|AR| = 2|MR|$.

A: $\angle QAR \simeq \angle PMR$, since they are alternate interior angles between the two parallel lines ℓ_P and ℓ_Q.

S: Q, the orthocenter of $\triangle ABC$, is also the circumcenter of another triangle $\triangle abc$. This triangle is similar to $\triangle ABC$, but twice as big. That means that the distance from Q, the circumcenter of $\triangle abc$ to side bc is double the distance from P, the circumcenter of $\triangle ABC$, to side BC (it was an exercise at the end of the last lesson to show that distances from centers are scaled proportionally by a similarity– if you skipped that exercise then, you should do it now, at least for this one case). In short, $|AQ| = 2|MP|$.

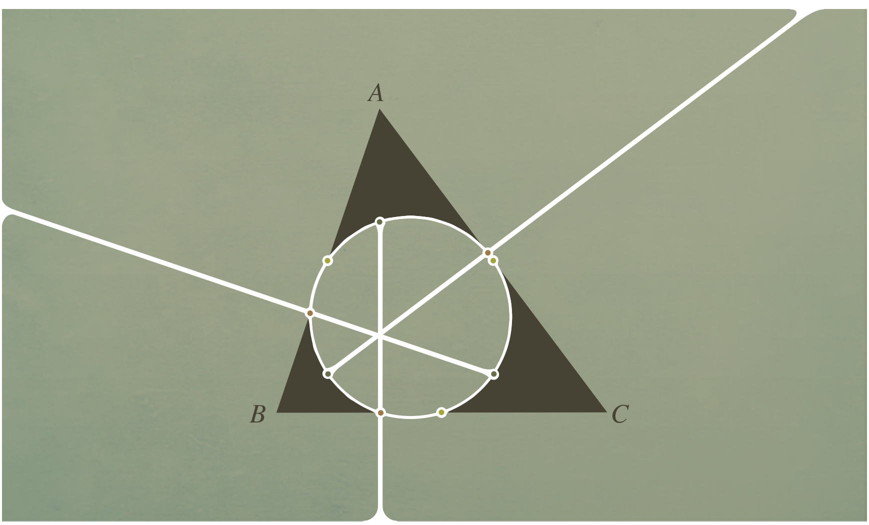
By S·A·S similarity, then, $\triangle AQR \sim \triangle MPR$. That means $\angle PRM$ is congruent to $\angle QRA$. The supplement of $\angle PRA$ is $\angle PRM$, so $\angle PRM$ must also be the supplement of $\angle QRA$. Therefore P, Q, and R are colinear. $\square$

The nine point circle

While only three points are needed to define a unique circle, the next result lists nine points associated with any triangle that are always on one circle. Six of the points were identified by Feuerbach (and for this reason the circle sometimes bears his name). Several more beyond the traditional nine have been found since. If you are interested in the development of this theorem, there is a brief history in *Geometry Revisited* by Coxeter and Greitzer [1].

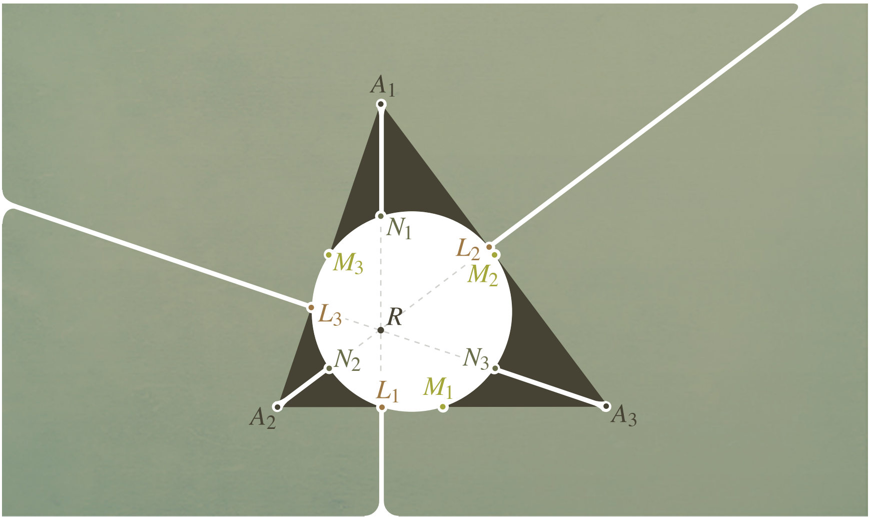
THM: THE NINE POINT CIRCLE

For any triangle, the following nine points all lie on the same circle: (1) the feet of the three altitudes, (2) the midpoints of the three sides, and (3) the midpoints of the three segments connecting the orthocenter to the each vertex. This circle is the *nine point circle* associated with that triangle.



This is a relatively long proof, and I would ask that you make sure you are aware of two key results that will play pivotal roles along the way.

1. Thales' Theorem: A triangle $\triangle ABC$ has a right angle at C if and only if C is on the circle with diameter AB.
2. The diagonals of a parallelogram bisect one another.



Proof. Given the triangle $\triangle A_1A_2A_3$ with orthocenter R, label the following nine points:

> L_i, the foot of the altitude which passes through A_i,
> M_i, the midpoint of the side that is opposite A_i,
> N_i, the midpoint of the segment A_iR.

The proof that I give here is based upon a key fact that is *not* mentioned in the statement of the theorem– that the segments M_iN_i are diameters of the nine point circle. We will take $\mathcal{C}$, the circle with diameter M_1N_1 and show that the remaining seven points are all on it. Allow me a moment to outline the strategy. First, we will show that the four angles

$$\angle M_1M_2N_1 \quad \angle M_1N_2N_1 \quad \angle M_1M_3N_1 \quad \angle M_1N_3N_1$$

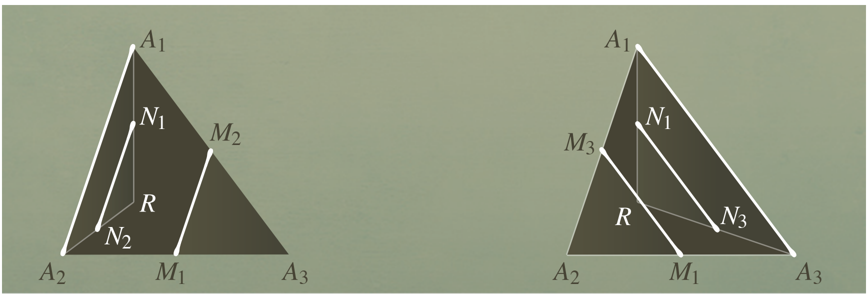
are right angles. By Thales' Theorem, that will place each of the points M_2, M_3, N_2, and N_3 on $\mathcal{C}$. Second, we will show that M_2N_2 and M_3N_3 are in fact diameters of $\mathcal{C}$. Third and finally, we will show that each $\angle M_iL_iN_i$ is a right angle, thereby placing the L_i on $\mathcal{C}$.

Lines that are parallel.
We need to prove several sets of lines are parallel to one another. The key in each case is S·A·S triangle similarity, and the argument for that similarity is the same each time. Let me just show you with the first one, and then I will leave out the details on all that follow.

Observe in triangles $\triangle A_3M_1M_2$ and $\triangle A_3A_2A_1$ that

$$|A_3M_2| = \tfrac{1}{2}|A_3A_1| \quad \angle A_3 = \angle A_3 \quad |A_3M_1| = \tfrac{1}{2}|A_3A_2|.$$

By the S·A·S similarity theorem, then, they are similar. In particular, the corresponding angles $\angle M_2$ and $\angle A_1$ in those triangles are congruent. According to the Alternate Interior Angle Theorem, M_1M_2 and A_1A_2 must be parallel. Let's employ that same argument many more times.

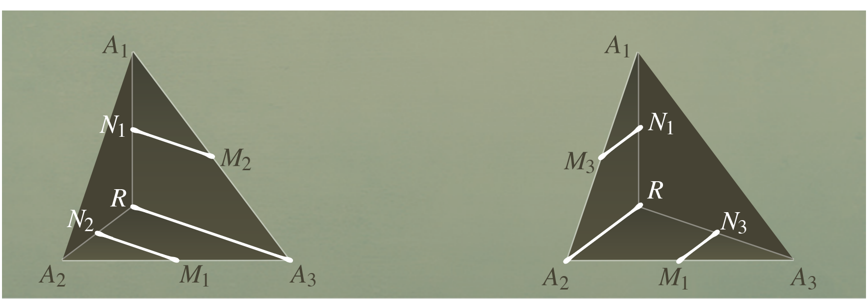


$\triangle A_3M_1M_2 \sim \triangle A_3A_2A_1$
$\implies M_1M_2 \parallel A_1A_2$

$\triangle RN_1N_2 \sim \triangle RA_1A_2$
$\implies N_1N_2 \parallel A_1A_2$

$\triangle A_2M_1M_3 \sim \triangle A_2A_3A_1$
$\implies M_1M_3 \parallel A_1A_3$

$\triangle RN_1N_3 \sim \triangle RA_1A_3$
$\implies N_1N_3 \parallel A_1A_3$

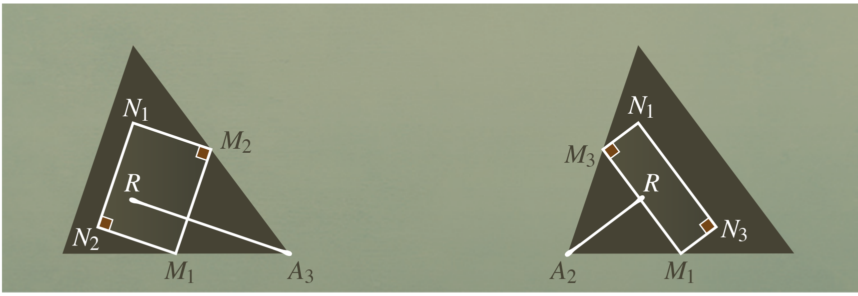


$\triangle A_1N_1M_2 \sim \triangle A_1RA_3$
$\implies N_1M_2 \parallel A_3R$

$\triangle A_2M_1N_2 \sim \triangle A_2A_3R$
$\implies M_1N_2 \parallel A_3R$

$\triangle A_1M_3N_1 \sim \triangle A_1A_2R$
$\implies M_3N_1 \parallel A_2R$

$\triangle A_3M_1N_3 \sim \triangle A_3A_2R$
$\implies M_1N_3 \parallel A_2R$



Angles that are right.

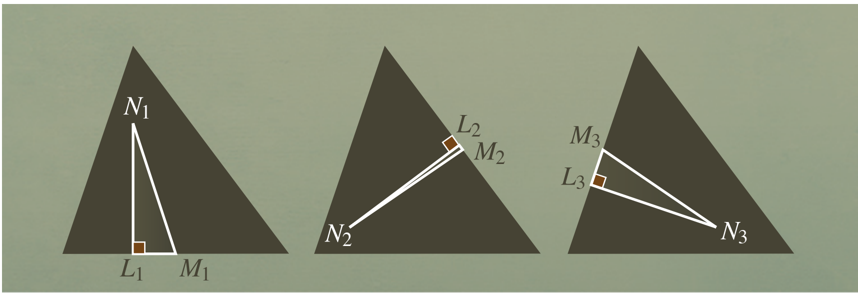
Now A_3R is a portion of the altitude perpendicular to A_1A_2. That means the first set of parallel lines are all perpendicular to the second set of parallel lines. Therefore M_1M_2 and M_2N_1 are perpendicular, so $\angle M_1M_2N_1$ is a right angle; and N_1N_2 and N_2M_1 are perpendicular, so $\angle M_1N_2N_1$ is a right angle. By Thales' Theorem, both M_2 and N_2 are on $\mathcal{C}$.

Similarly, segment A_2R is perpendicular to A_1A_3 (an altitude and a base), so M_1M_3 and M_3N_1 are perpendicular, and so $\angle M_1M_3N_1$ is a right angle. Likewise, N_1N_3 and N_3M_1 are perpendicular, so $\angle M_1N_3N_1$ is a right angle. Again Thales' Theorem tells us that M_3 and N_3 are on $\mathcal{C}$.

Segments that are diameters.
We have all the M's and N's placed on $\mathcal{C}$ now, but we aren't done with them just yet. Remeber that M_1N_1 is a diameter of $\mathcal{C}$. From that, it is just a quick hop to show that L_1 is also on $\mathcal{C}$. It would be nice to do the same for L_2 and L_3, but in order to do that we will have to know that M_2N_2 and M_3N_3 are also diameters. Based upon our work above,

$$M_1M_2 \parallel N_1N_2 \quad \& \quad M_1N_2 \parallel M_2N_1$$

That makes $\square M_1M_2N_1N_2$ a parallelogram (in fact it is a rectangle). Its two diagonals, M_1N_1 and M_2N_2 must bisect each other. In other words, M_2N_2 crosses M_1N_1 at its midpoint. Well, the midpoint of M_1N_1 is the center of $\mathcal{C}$. That means that M_2N_2 passes through the center of $\mathcal{C}$, and that makes it a diameter. The same argument works for M_3N_3. The parallelogram is $\square M_1M_3N_1N_3$ with bisecting diagonals M_1N_1 and M_3N_3.

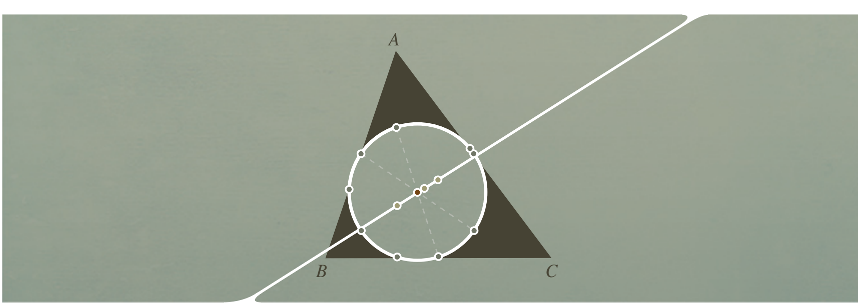


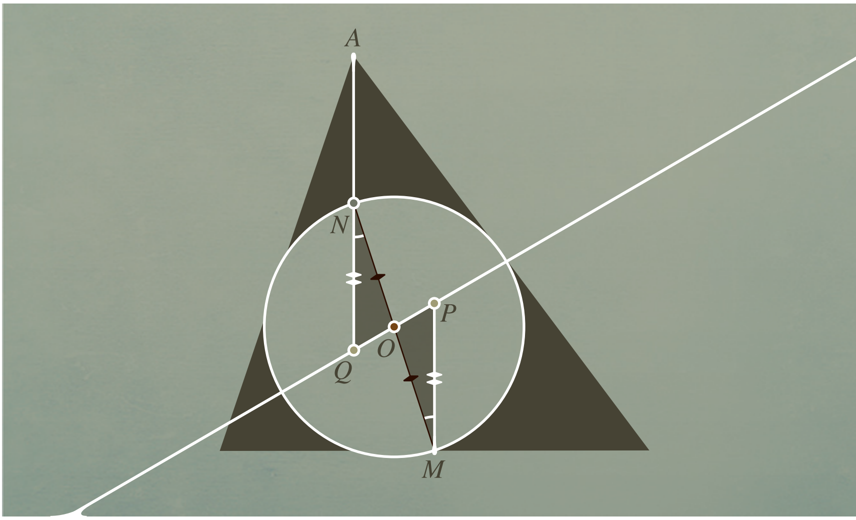
More angles that are right.
All three of M_1N_1, M_2N_2, and M_3N_3 are diameters of $\mathcal{C}$. All three of $\angle M_1L_1N_1$, $\angle M_2L_2N_2$ and $M_3L_3N_3$ are formed by the intersection of an altitude and a base, and so are right angles. Therefore, by Thales' Theorem, all three of L_1, L_2 and L_3 are on $\mathcal{C}$. $\square$

The center of the nine point circle

The third result of this lesson ties together the previous two.

THM
The center of the nine point circle is on the Euler line.





Proof. This proof nicely weaves together a lot of what we have developed over the last two lessons. On $\triangle ABC$, label the circumcenter P and the orthocenter Q. Then $\overleftrightarrow{PQ}$ is the Euler line. Label the center of the nine point circle as O. Our last proof hinged upon a diameter of the nine point circle. Let's recycle some of that– if M is the midpoint of BC and N is the midpoint of QA, then MN is a diameter of the nine point circle. Now this proof really boils down to a single triangle congruence– we need to show that $\triangle ONQ$ and $\triangle OMP$ are congruent.

S: $ON \simeq OM$. The center O of the nine point circle bisects the diameter MN.

A: $\angle M \simeq \angle N$. These are alternate interior angles between two parallel lines, the altitude and bisector perpendicular to BC.

S: $NQ \simeq MP$. In the Euler line proof we saw that $|AQ| = 2|MP|$. Well, $|NQ| = \frac{1}{2}|AQ|$, so $|NQ| = |MP|$.

By S·A·S, the triangles $\triangle ONQ$ and $\triangle OMP$ are congruent, and in particular $\angle QON \simeq \angle POM$. Since $\angle NOP$ is supplementary to $\angle POM$, it must also be supplementary to $\angle QON$. Therefore Q, O, and P are colinear, and so O is on the Euler line. $\square$

Exercises

1. Consider a triangle $\triangle ABC$. Let D and E be the feet of the altitudes on the sides AC and BC. Prove that there is a circle which passes through the points A, B, D, and E.

2. Under what conditions does the incenter lie on the Euler line?

3. Consider an isosceles triangle $\triangle ABC$ with $AB \simeq AC$. Let D be a point on the arc between B and C of the circumscribing circle. Show that DA bisects the angle $\angle BDC$.

4. Let P be a point on the circumcircle of triangle $\triangle ABC$. Let L be the foot of the perpendicular from P to AB, M be the foot of the perpendicular from P to AC, and N be the foot of the perpendicular from P to BC. Show that L, M, and N are collinear. This line is called a *Simson line*. Hint: look for cyclic quadrilaterals and use the fact that opposite angles in a cyclic quadrilateral are congruent.

References

[1] H.S.M. Coxeter and Samuel L. Greitzer. *Geometry Revisited*. Random House, New York, 1st edition, 1967.