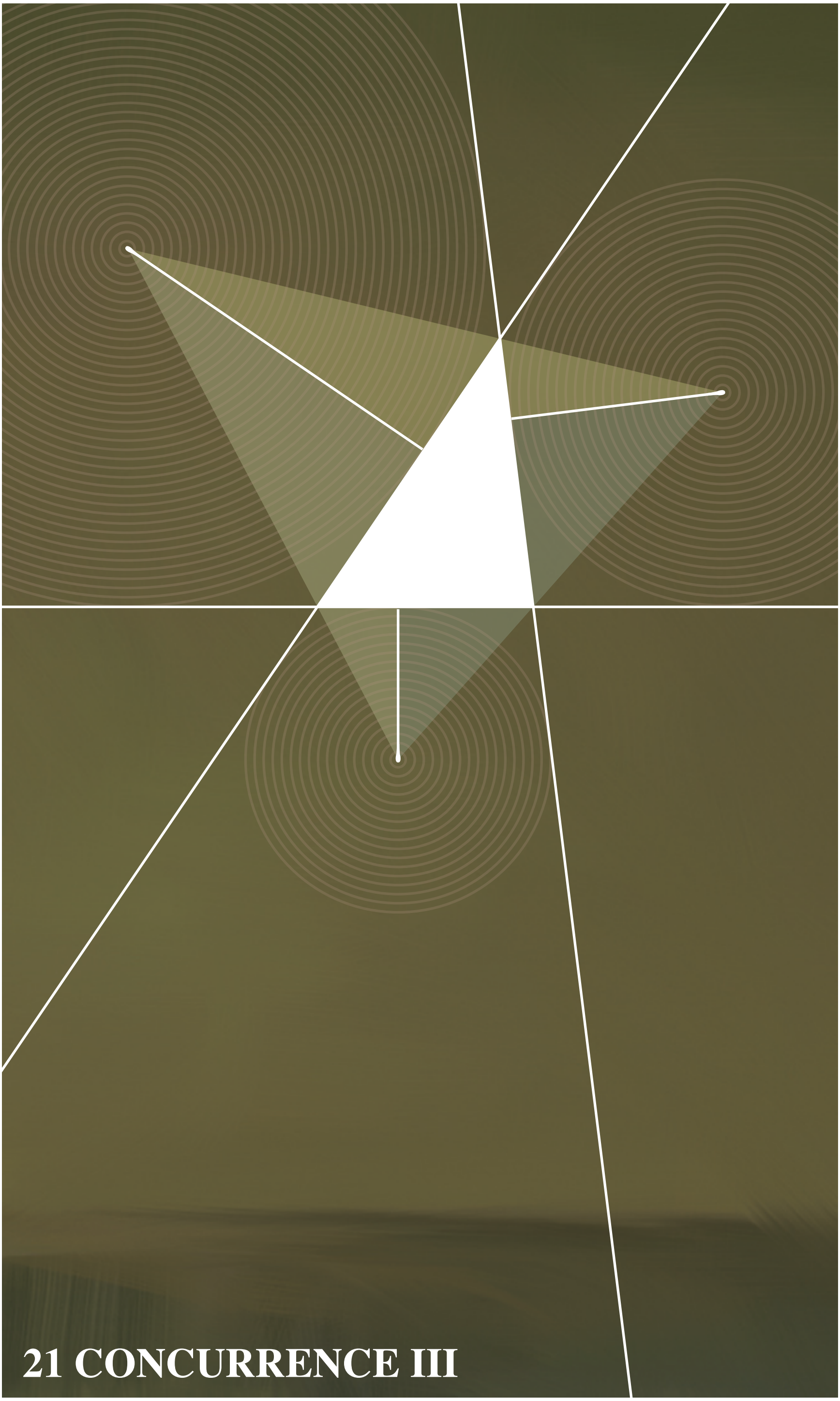
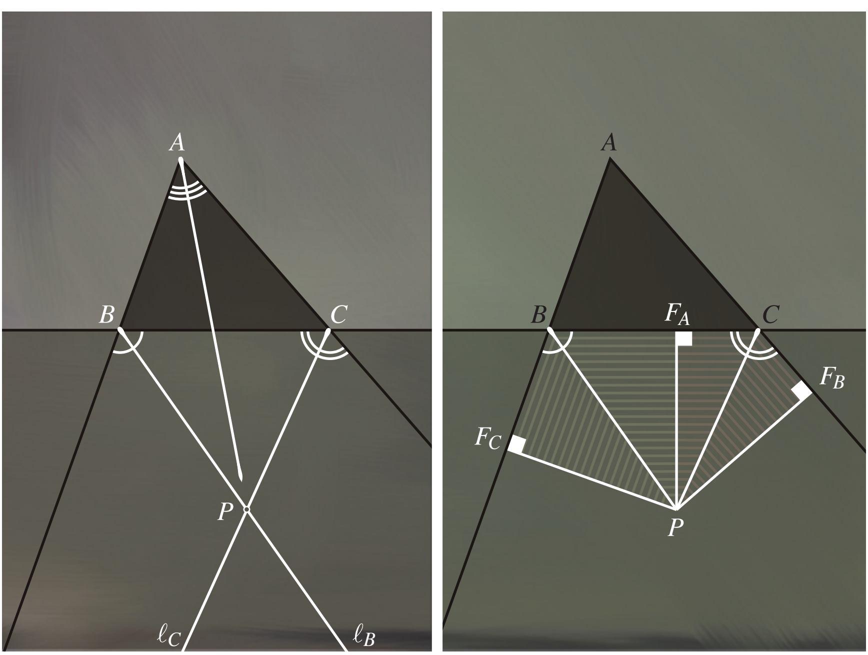
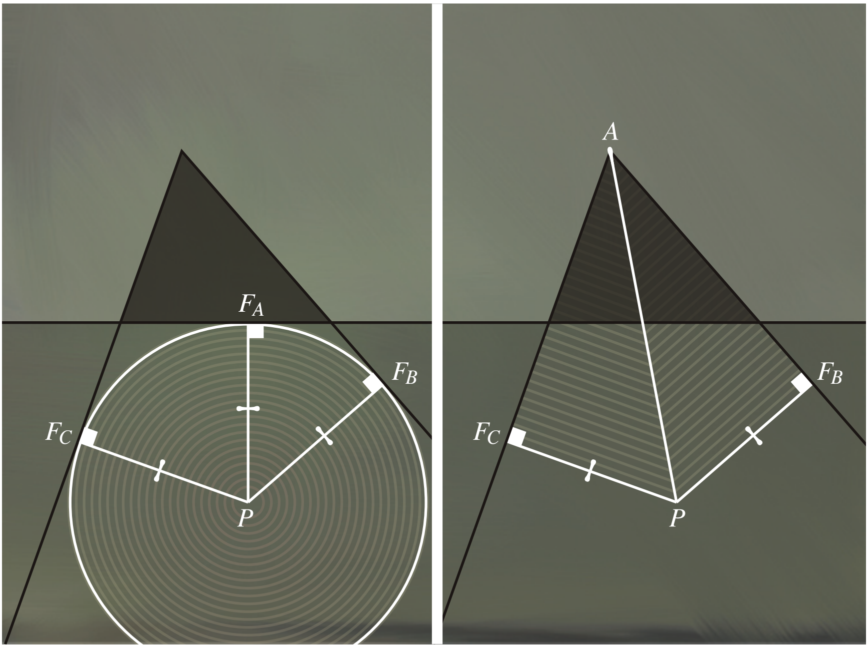

21 CONCURRENCE III

Excenters and excircles

In the first lesson on concurrence, we saw that the bisectors of the interior angles of a triangle concur at the incenter. If you did the exercise in the last lesson dealing with the orthic triangle then you may have noticed something else– that the sides of the original triangle are the bisectors of the exterior angles of the orthic triangle. I want to lead off this last lesson on concurrence with another result that connects interior and exterior angle bisectors.

THM: EXCENTERS
The exterior angle bisectors at two vertices of a triangle and the interior angle bisector at the third vertex of that triangle intersect at one point.





Proof. Let ℓ_B and ℓ_C be the lines bisecting the exterior angles at vertices B and C of $\triangle ABC$. They must intersect. Label the point of intersection as P. Now we need to show that the interior angle bisector at A must also cross through P, but we are going to have to label a few more points to get there. Let F_A, F_B, and F_C be the feet of the perpendiculars through P to each of the sides BC, AC, and AB, respectively. Then, by A·A·S,

$$\triangle PF_AC \simeq \triangle PF_BC \quad \triangle PF_AB \simeq \triangle PF_CB.$$

Therefore $PF_A \simeq PF_B \simeq PF_C$. Here you may notice a parallel with the previous discussion of the incenter– P, like the incenter, is equidistant from the lines containing the three sides of the triangle. By H·L right triangle congruence, $\triangle PF_CA \simeq \triangle PF_BA$. In particular, $\angle PAF_C \simeq \angle PAF_B$ and so P is on the bisector of angle A. □

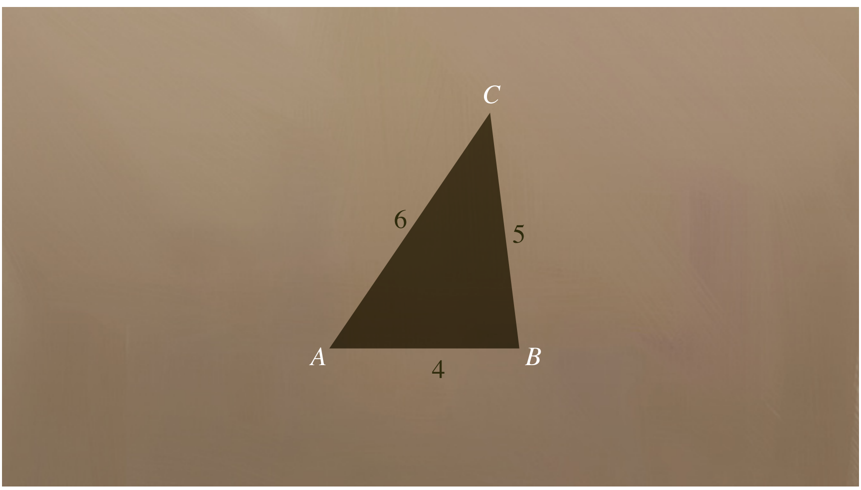
There are three such points of concurrence. They are called the *excenters* of the triangle. Since each is equidistant from the three lines containing the sides of the triangle, each is the center of a circle tangent to those three lines. Those circles are called the *excircles* of the triangle.

Ceva's Theorem

By now, you should have seen enough concurrence theorems and enough of their proofs to have some sense of how they work. Most of them ultimately turn on a few hidden triangles that are congruent or similar. Take, for example, the concurrence of the medians. The proof of that concurrence required a 2 : 1 ratio of triangles. What about other triples of segments that connect the vertices of a triangle to their respective opposite sides? What we need is a computation that will discriminate between triples of segments that do concur and triples of segments that do not.

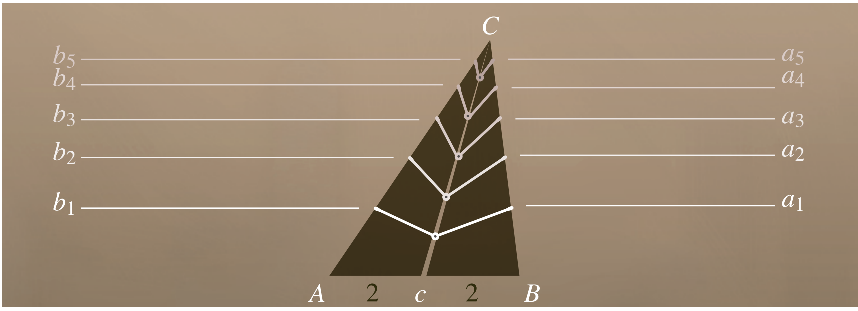
Let's experiment. Here is a triangle $\triangle ABC$ with sides of length four, five, and six.

$$|AB| = 4 \quad |BC| = 5 \quad |AC| = 6.$$



As an easy initial case, let's say that one of the three segments, say Cc, is a median (in other words, that c is the midpoint of AB). Now work backwards. Say that the triple of segments in question are concurrent. That concurrence could happen anywhere along Cc, so I have chosen five points P_i to serve as our sample points of concurrence. Once those points of concurrence have been chosen, that determines the other two segments– one passes through A and P_i, the other through B and P_i. I am interested in where those segments cut the sides of $\triangle ABC$. Label:

b_i: the intersection of BP_i and AC
a_i: the intersection of AP_i and BC



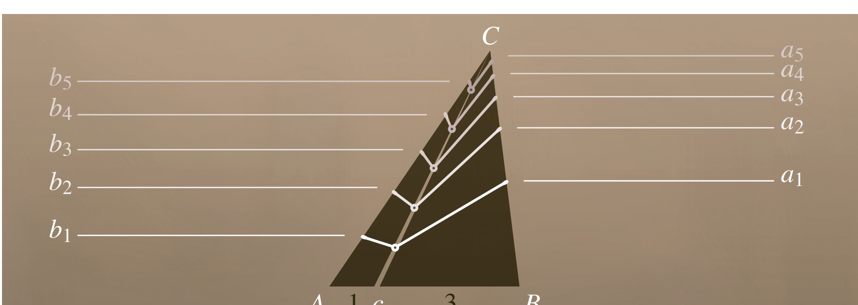
Here are the measurements (two decimal place accuracy):

i :	1	2	3	4	5
$\lvert Ab_i \rvert$	1.71	3.00	4.00	4.80	5.45
$\lvert Cb_i \rvert$	4.29	3.00	2.00	1.20	0.55
$\lvert Ba_i \rvert$	1.43	2.50	3.33	4.00	4.55
$\lvert Ca_i \rvert$	3.57	2.50	1.67	1.00	0.45

Out of all of that it may be difficult to see a useful pattern, but compare the ratios of the sides $|Ab_i|/|Cb_i|$ and $|Ba_i|/|Ca_i|$ (after all, similarity is all about ratios).

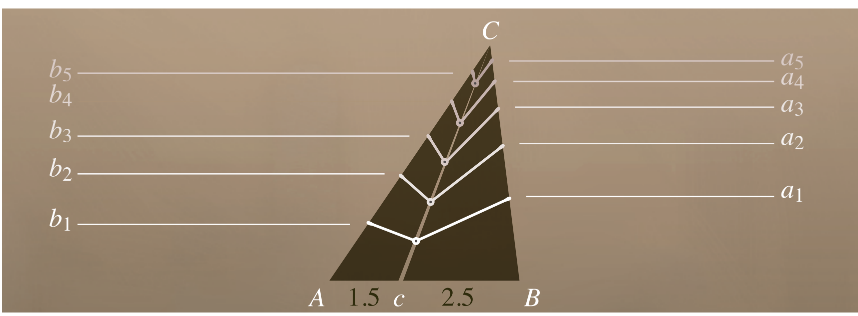
i :	1	2	3	4	5
$\lvert Ab_i \rvert / \lvert Cb_i \rvert$	0.40	1.00	2.00	4.00	10.00
$\lvert Ba_i \rvert / \lvert Ca_i \rvert$	0.40	1.00	2.00	4.00	10.00

They are the same! Let's not jump the gun though– what if Cc isn't a median? For instance, let's reposition c so that it is a distance of one from A and three from B.



i:	1	2	3	4	5
$\lvert Ab_i \rvert$	1.26	2.40	3.43	4.36	5.22
$\lvert Cb_i \rvert$	4.74	3.60	2.57	1.64	0.78
$\lvert Ba_i \rvert$	2.22	3.33	4.00	4.45	4.76
$\lvert Ca_i \rvert$	2.78	1.67	1.00	0.55	0.24
$\lvert Ab_i \rvert / \lvert Cb_i \rvert$	0.27	0.67	1.33	2.67	6.67
$\lvert Ba_i \rvert / \lvert Ca_i \rvert$	0.80	2.00	4.00	8.02	20.12

The ratios are not the same. Look carefully, though– the ratios $|Ba_i|/|Ca_i|$ are always three times the corresponding ratios $|Ab_i|/|Cb_i|$ (other than a bit of round-off error). Interestingly, that is the same as the ratio $|Bc|/|Ac|$. Let's do one more example, with $|Ac| = 1.5$ and $|Bc| = 2.5$.

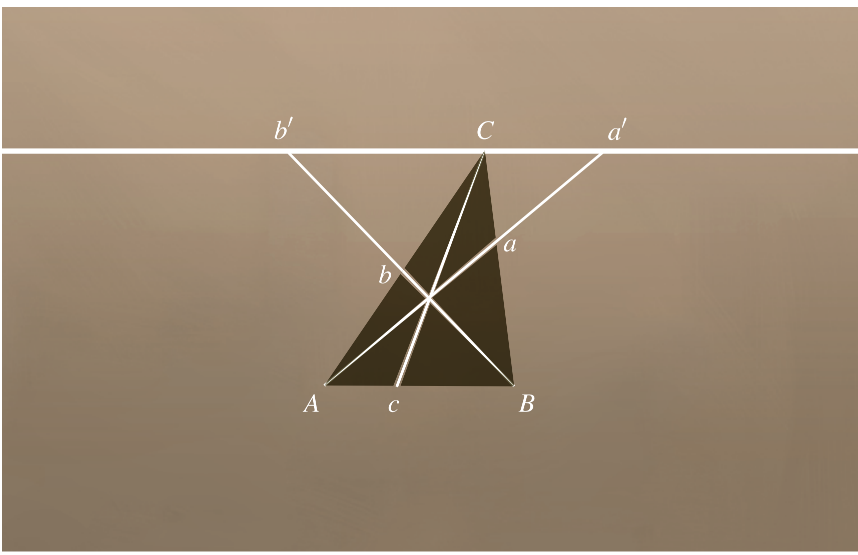


i:	1	2	3	4	5
$\lvert Ab_i \rvert$	1.45	2.67	3.69	4.57	5.33
$\lvert Cb_i \rvert$	4.55	3.33	2.31	1.43	0.67
$\lvert Ba_i \rvert$	1.74	2.86	3.64	4.21	4.65
$\lvert Ca_i \rvert$	3.26	2.14	1.36	0.79	0.35
$\lvert Ab_i \rvert / \lvert Cb_i \rvert$	0.32	0.80	1.60	3.20	8.00
$\lvert Ba_i \rvert / \lvert Ca_i \rvert$	0.53	1.33	2.66	5.34	13.33

Once again, the ratios $|Ab_i|/|Cb_i|$ all hover about 1.67, right at the ratio $|Bc|/|Ac|$. What we have stumbled across is called Ceva's Theorem, but it is typically given a bit more symmetrical presentation.

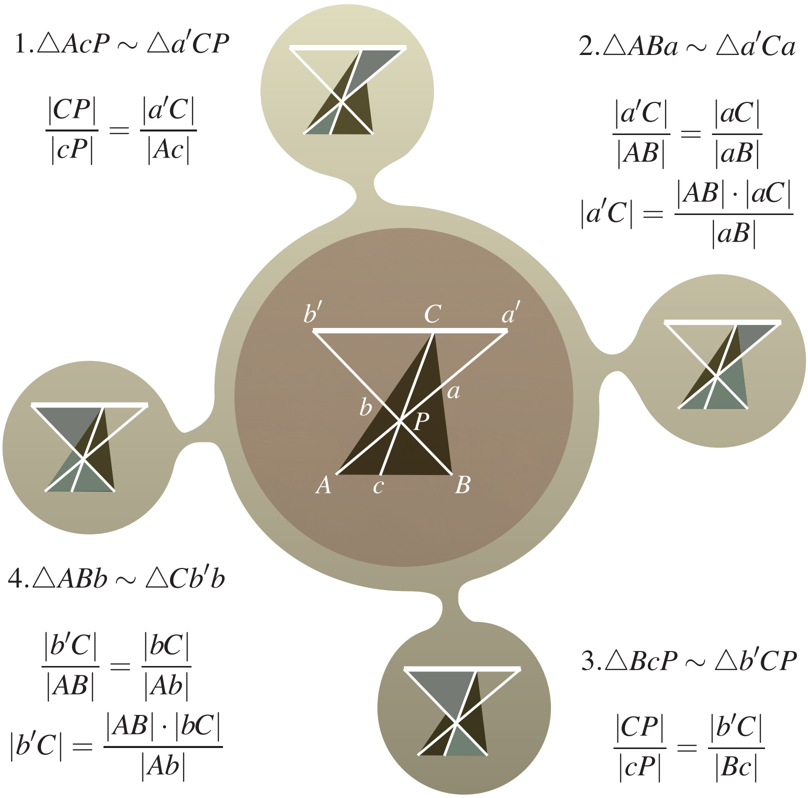
CEVA'S THEOREM
Three segments Aa, Bb, and Cc, that connect the vertices of $\triangle ABC$ to their respective opposite sides, are concurrent if and only if

$$\frac{|Ab|}{|bC|} \cdot \frac{|Ca|}{|aB|} \cdot \frac{|Bc|}{|cA|} = 1.$$



Proof. $\Longrightarrow$ Similar triangles anchor this proof. To get to those similar triangles, though, we need to extend the illustration a bit. Assume that Aa, Bb, and Cc concur at a point P. Draw out the line which passes through C and is parallel to AB; then extend Aa and Bb so that they intersect this line. Mark those intersection points as d and b' respectively. We need to look at four pairs of similar triangles.

They are:



1. $\triangle AcP \sim \triangle a'CP$

$$\frac{|CP|}{|cP|} = \frac{|a'C|}{|Ac|}$$

2. $\triangle ABa \sim \triangle a'Ca$

$$\frac{|a'C|}{|AB|} = \frac{|aC|}{|aB|}$$

$$|a'C| = \frac{|AB| \cdot |aC|}{|aB|}$$

3. $\triangle BcP \sim \triangle b'CP$

$$\frac{|CP|}{|cP|} = \frac{|b'C|}{|Bc|}$$

4. $\triangle ABb \sim \triangle Cb'b$

$$\frac{|b'C|}{|AB|} = \frac{|bC|}{|Ab|}$$

$$|b'C| = \frac{|AB| \cdot |bC|}{|Ab|}$$

Plug the second equation into the first

$$\frac{|CP|}{|cP|} = \frac{|AB| \cdot |aC|}{|aB| \cdot |Ac|}$$

and the fourth into the third

$$\frac{|CP|}{|cP|} = \frac{|AB| \cdot |bC|}{|Ab| \cdot |BC|}$$

Set these two equations equal and simplify

$$\frac{|AB| \cdot |aC|}{|aB| \cdot |Ac|} = \frac{|AB| \cdot |bC|}{|Ab| \cdot |BC|} \implies \frac{|Ab|}{|bC|} \cdot \frac{|Ca|}{|aB|} \cdot \frac{|Bc|}{|cA|} = 1.$$

$\Longleftarrow$ A similar tactic works for the other direction. For this part, we are going to assume the equation

$$\frac{|Ab|}{|bC|}\cdot\frac{|Ca|}{|aB|}\cdot\frac{|Bc|}{|cA|}=1,$$

and show that $Aa, Bb,$ and Cc are concurrent. Label

P: the intersection of Aa and Cc
Q: the intersection of Bb and Cc.

In order for all three segments to concur, P and Q will actually have to be the same point. We can show that they are by computing the ratios $|AP|/|aP|$ and $|AQ|/|aQ|$ and seeing that they are equal. That will mean that P and Q have to be the same distance down the segment Aa from A, and thus guarantee that they are the same. Again with the similar triangles:

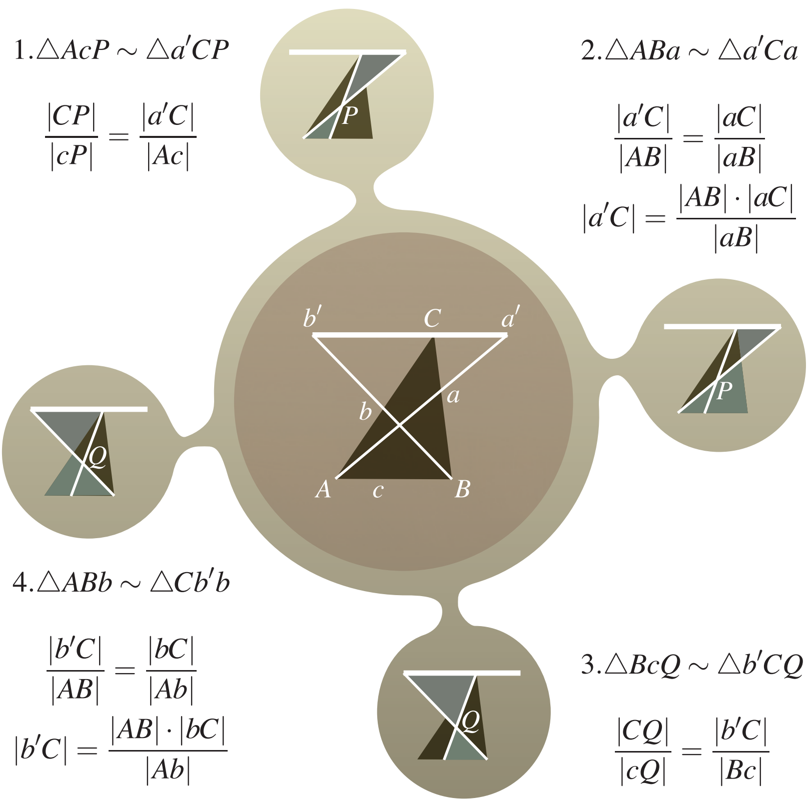
1. $\triangle AcP \sim \triangle a'CP$

$$\frac{|CP|}{|cP|}=\frac{|a'C|}{|Ac|}$$

2. $\triangle ABa \sim \triangle a'Ca$

$$\frac{|a'C|}{|AB|}=\frac{|aC|}{|aB|}$$

$$|a'C|=\frac{|AB|\cdot|aC|}{|aB|}$$



4. $\triangle ABb \sim \triangle Cb'b$

$$\frac{|b'C|}{|AB|}=\frac{|bC|}{|Ab|}$$

$$|b'C|=\frac{|AB|\cdot|bC|}{|Ab|}$$

3. $\triangle BcQ \sim \triangle b'CQ$

$$\frac{|CQ|}{|cQ|}=\frac{|b'C|}{|Bc|}$$

Plug the second equation into the first

$$\frac{|CP|}{|cP|} = \frac{|aC| \cdot |AB|}{|aB| \cdot |Ac|}$$

and the fourth equation into the third

$$\frac{|CQ|}{|cQ|} = \frac{|AB| \cdot |bC|}{|Ab| \cdot |Bc|}$$

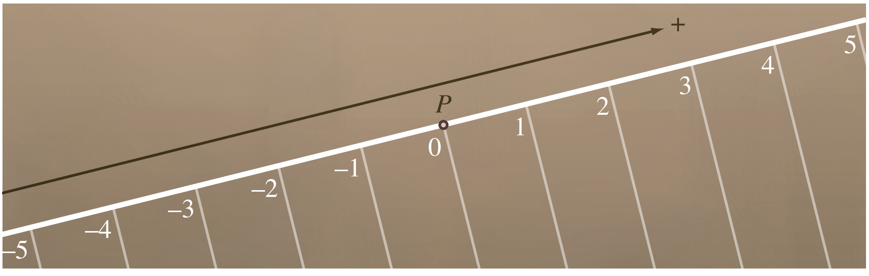
Now divide and simplify

$$\frac{|CP|}{|cP|} \bigg/ \frac{|CQ|}{|cQ|} = \frac{|aC| \cdot |AB| \cdot |Ab| \cdot |Bc|}{|aB| \cdot |Ac| \cdot |AB| \cdot |bC|} = \frac{|Ab|}{|bC|} \cdot \frac{|Ca|}{|aB|} \cdot \frac{|Bc|}{|cA|} = 1.$$

Therefore $|AP|/|aP| = |AQ|/|aQ|$, so $P = Q$. $\square$

Ceva's Theorem is great for concurrences inside the triangle, but we have seen that concurrences can happen outside the triangle as well (such as the orthocenter of an obtuse triangle). Will this calculation still tell us about those concurrences? Well, not quite. If the three lines concur, then the calculation will still be one, but now the calculation can mislead– it is possible to calculate one when the lines do not concur. If you look back at the proof, you can see the problem. If P and Q are on the opposite side of a, then the ratios $|AP|/|aP|$ and $|AQ|/|aQ|$ could be the same even though $P \neq Q$. There is a way to repair this, though. The key is "signed distance". We assign to each of the three lines containing a side of the triangle a direction (saying this way is positive, this way is negative). For two points A and B on one of those lines, the signed distance is defined as

$$[AB] = \begin{cases} |AB| & \text{if the ray } AB\rightarrow \text{ points in the positive direction} \\ -|AB| & \text{if the ray } AB\rightarrow \text{ points in the negative direction.} \end{cases}$$



Signed distance from P. The sign is determined by a choice of direction.

This simple modification is all that is needed to extend Ceva's Theorem

CEVA'S THEOREM (EXTENDED VERSION)

Three lines Aa, Bb, and Cc, that connect the vertices of $\triangle ABC$ to the lines containing their respective opposite sides, are concurrent if and only if

$$\frac{[Ab]}{[bC]} \cdot \frac{[Ca]}{[aB]} \cdot \frac{[Bc]}{[cA]} = 1.$$

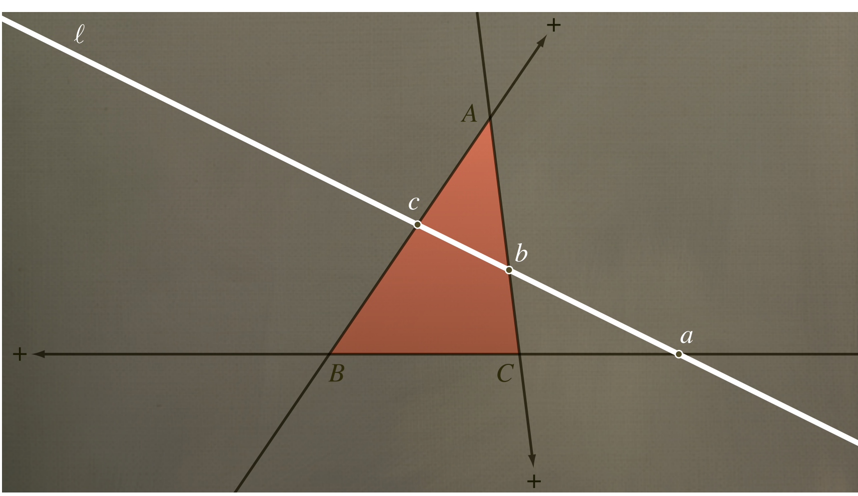
Menelaus's Theorem

Ceva's Theorem is one of a pair– the other half is its projective dual, Menelaus's Theorem. We are not going to look at projective geometry in this book, but one of its key underlying concepts is that at the level of incidence, there is a duality between points and lines. For some very fundamental results, this duality allows the roles of the two to be interchanged.

MENELAUS'S THEOREM

For a triangle $\triangle ABC$, and points a on $\leftarrow BC \rightarrow$, b on $\leftarrow AC \rightarrow$, and c on $\leftarrow AB \rightarrow$, a, b, and c are colinear if and only if

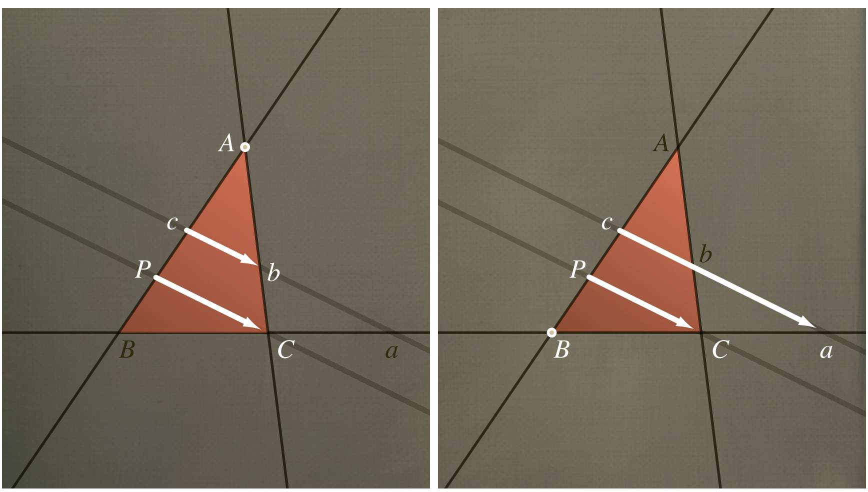
$$\frac{[Ab]}{[bC]} \cdot \frac{[Ca]}{[aB]} \cdot \frac{[Bc]}{[cA]} = -1.$$



Proof. $\Longrightarrow$ Suppose that a, b, and c all lie along a line ℓ. The requirement that a, b, and c all be distinct prohibits any of the three intersections from occurring at a vertex. According to Pasch's Lemma, then, ℓ will intersect two sides of the triangle, or it will miss all three sides entirely. Either way, it has to miss one of the sides. Let's say that missed side is BC. There are two ways this can happen:

1. ℓ intersects line BC on the opposite side of B from C
2. ℓ intersects line BC on the opposite side of C from B

The two scenarios will play out very similarly, so let's just look at the second one. Draw the line through C parallel to ℓ. Label its intersection with AB as P. That sets up some useful parallel projections.



From AB to AC:

$$A \mapsto A \quad c \mapsto b \quad P \mapsto C.$$

Comparing ratios,

$$\frac{|cP|}{|bC|} = \frac{|Ac|}{|Ab|}$$

and so

$$|cP| = \frac{|Ac|}{|Ab|} \cdot |bC|.$$

From AB to BC:

$$B \mapsto B \quad c \mapsto a \quad P \mapsto C.$$

Comparing ratios,

$$\frac{|cP|}{|aC|} = \frac{|Bc|}{|Ba|}$$

and so

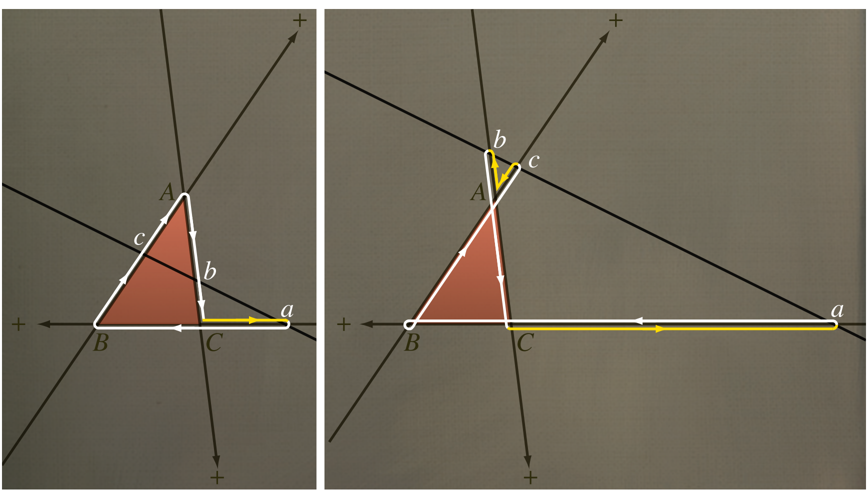
$$|cP| = \frac{|Bc|}{|Ba|} \cdot |aC|.$$

Just divide the second $|cP|$ by the first $|cP|$ to get

$$1 = \frac{|cP|}{|cP|} = \frac{|Ab|\cdot|aC|\cdot|Bc|}{|Ac|\cdot|bC|\cdot|Ba|} = \frac{|Ab|}{|bC|}\cdot\frac{|Ca|}{|aB|}\cdot\frac{|Bc|}{|cA|}.$$

That's close, but we are after an equation that calls for *signed* distance. So orient the three lines of the triangle so that $AC\rightarrow$, $CB\rightarrow$, and $BA\rightarrow$ all point in the positive direction (any other orientation will flip pairs of signs that will cancel each other out). With this orientation, if ℓ intersects two sides of the triangle, then all the signed distances involved are positive except $[Ca] = -|Ca|$. If ℓ misses all three sides of the triangle, then three of the signed distances are positive, but three are not:

$$[Ab] = -|Ab| \quad [Ca] = -|Ca| \quad [cA] = -|cA|.$$

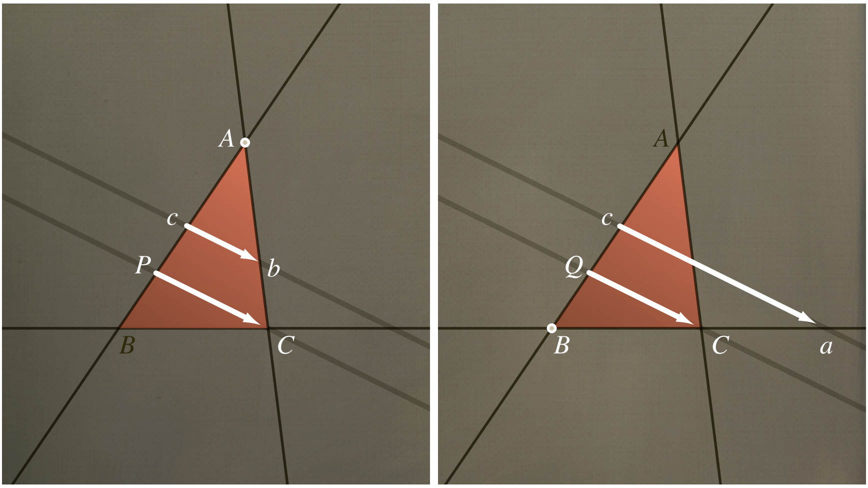


Either way, an odd number of signs are changed, so

$$\frac{[Ab]}{[bC]}\frac{[Ca]}{[aB]}\frac{[Bc]}{[cA]} = -1.$$

$\Longleftarrow$ Let's turn the argument around to prove the converse. Suppose that

$$\frac{[Ab]}{[bC]}\cdot\frac{[Ca]}{[aB]}\cdot\frac{[Bc]}{[cA]} = -1.$$



Draw the line from C that is parallel to bc and label its intersection with AB as P. There is a parallel projection from AB to AC so that

$$A \mapsto A \quad c \mapsto b \quad P \mapsto C$$

and therefore

$$\frac{|cP|}{|Ac|} = \frac{|Cb|}{|bA|}.$$

Draw the line from C that is parallel to ac, and label its intersection with AB as Q. There is a parallel projection from AB to BC so that

$$B \mapsto B \quad c \mapsto a \quad Q \mapsto C$$

and therefore

$$\frac{|cQ|}{|cB|} = \frac{|Ca|}{|Ba|}$$

Now solve those equations for $|cP|$ and $|cQ|$, and divide to get

$$\frac{[cQ]}{[cP]} = \frac{[bA]\cdot[Ca]\cdot[cB]}{[Cb]\cdot[Ac]\cdot[Ba]} = -\frac{[Ab]}{[bC]}\cdot\frac{[Ca]}{[aB]}\cdot\frac{[Bc]}{[cA]} = -(-1) = 1.$$

Both P and Q are the same distance from c along cC. That means they must be the same. $\square$

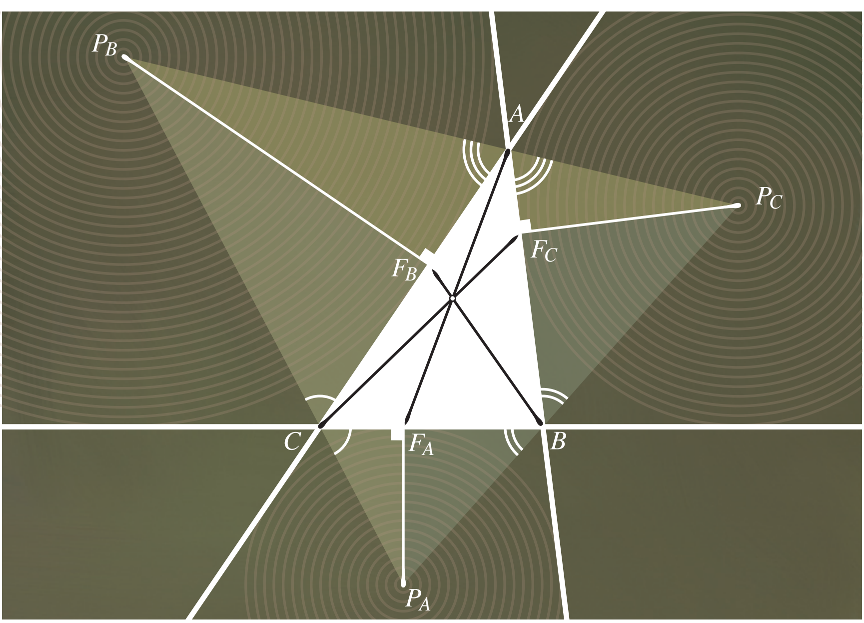
The Nagel point

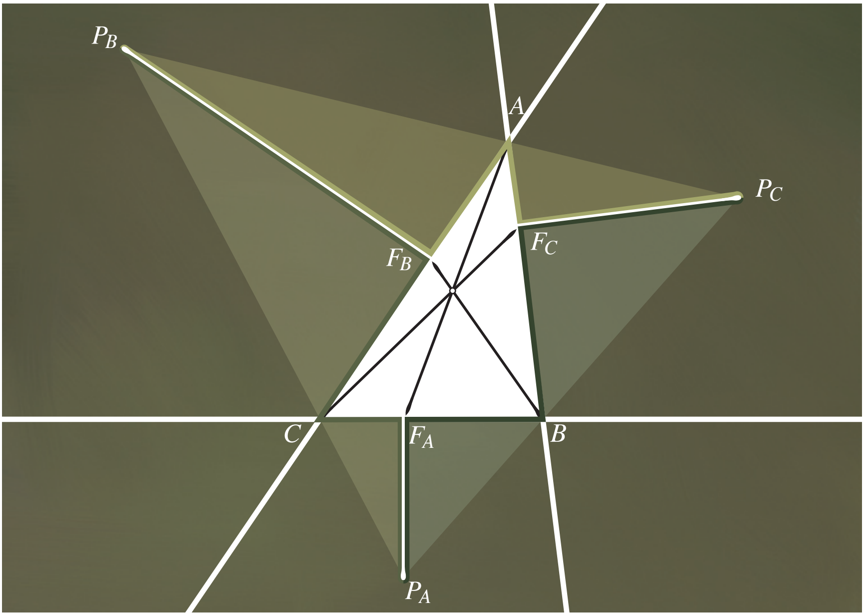
Back to excircles for one more concurrence, and this time we will use Ceva's Theorem to prove it.

THE NAGEL POINT
If $\mathcal{C}_A$, $\mathcal{C}_B$, and $\mathcal{C}_C$ are the three excircles of a triangle $\triangle ABC$ so that $\mathcal{C}_A$ is in the interior of $\angle A$, $\mathcal{C}_B$ is in the interior of $\angle B$, and $\mathcal{C}_C$ is in the interior of $\angle C$; and if F_A is the intersection of $\mathcal{C}_A$ with BC, F_B is the intersection of $\mathcal{C}_B$ with AC, and F_C is the intersection of $\mathcal{C}_C$ with AB; then the three segments AF_A, BF_B, and CF_C are concurrent. This point of concurrence is called the Nagel point.

Proof. This is actually pretty easy thanks to Ceva's Theorem. The key is similar triangles. Label P_A, the center of excircle $\mathcal{C}_A$, P_B, the center of excircle $\mathcal{C}_B$, and P_C, the center of excircles, $\mathcal{C}_C$. By A·A triangle similarity,

$$\triangle P_AF_AC \sim \triangle P_BF_BC$$
$$\triangle P_BF_BA \sim \triangle P_CF_CA$$
$$\triangle P_CF_CB \sim \triangle P_AF_AB.$$





Ceva's Theorem promises concurrence if we can show that

$$\frac{|AF_C|}{|F_CB|} \cdot \frac{|BF_A|}{|F_AC|} \cdot \frac{|CF_B|}{|F_BA|} = 1.$$

Those triangle similarities give some useful ratios to that end:

$$\frac{|AF_C|}{|AF_B|} = \frac{|P_CF_C|}{|P_BF_B|} \quad \frac{|BF_A|}{|BF_C|} = \frac{|P_AF_A|}{|P_CF_C|} \quad \frac{|CF_B|}{|CF_A|} = \frac{|P_BF_B|}{|P_AF_A|}.$$

So

$$\begin{aligned} \frac{|AF_C|}{|F_CB|} \frac{|BF_A|}{|F_AC|} \frac{|CF_B|}{|F_BA|} &= \frac{|AF_C|}{|AF_B|} \frac{|BF_A|}{|BF_C|} \frac{|CF_B|}{|CF_A|} \\ &= \frac{|P_CF_C|}{|P_BF_B|} \frac{|P_AF_A|}{|P_CF_C|} \frac{|P_BF_B|}{|P_AF_A|} \\ &= 1. \end{aligned}$$

By Ceva's Theorem, the three segments are concurrent. □

Exercises

1. Use Ceva's Theorem to prove that the medians of a triangle are concurrent.

2. Use Ceva's Theorem to prove that the orthocenters of a triangle are concurrent.

3. Give a compass and straight-edge construction of the three excircles and the nine-point circle of a given triangle. If your construction is accurate enough, you should notice that the excircles are all tangent to the nine-point circle (a result commonly called Feuerbach's Theorem).